Poisson Geometry and Normal Forms: A Guided Tour through Examples

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From Poisson Geometry to Quantum Fields on Noncommutative Spaces, Würzburg Autumn School

Lectures 4 and 5
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Lectures 4 and 5
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<td>$\Pi$</td>
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<tr>
<td>$\iota_{X_f}\omega = -df$</td>
<td>$X_f := \Pi(df, \cdot)$</td>
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<td>one symplectic leaf</td>
<td>a symplectic foliation</td>
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<td>$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_i} + \sum_{kl} \phi_{kl}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l}$</td>
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<td>$L_X \omega = 0$</td>
<td>$L_X \Pi = 0$</td>
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<td>$H^1_{DR}(M) = \frac{\text{symplectic v.f}}{\text{Hamiltonian v.f}}$</td>
<td>$? = \frac{\text{Poisson v.f}}{\text{Hamiltonian v.f}}$</td>
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<td>$H^k_{DR}(M)$ (cochains $\Omega^k(M))$</td>
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Plan for today

- Poisson cohomology computation kit.

- Integrable systems on Poisson manifolds (Topology).

- Integrable systems on Poisson manifolds (Geometry and normal forms). The case of $b$-Poisson manifolds.

- Applications.
Case of vector fields,
\[ A = \sum_i a_i \frac{\partial}{\partial x_i} \text{ and } B = \sum_i b_i \frac{\partial}{\partial x_i}. \]
Then
\[
[A, B] = \sum_i a_i \left( \sum_j \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} \right) - \sum_i b_i \left( \sum_j \frac{\partial a_j}{\partial x_i} \frac{\partial}{\partial x_j} \right)
\]

Re-denoting \( \frac{\partial}{\partial x_i} \) as \( \zeta_i \) ("odd coordinates ").
Then \( A = \sum_i a_i \zeta_i \) and \( B = \sum_i b_i \zeta_i \) and \( \zeta_i \zeta_j = -\zeta_j \zeta_i \) Now we can reinterpret the bracket as,
\[
[A, B] = \sum_i \frac{\partial A}{\partial \zeta_i} \frac{\partial B}{\partial x_i} - \sum_i \frac{\partial B}{\partial \zeta_i} \frac{\partial A}{\partial x_i}
\]
Schouten Bracket of multivector fields in local coordinates

We reproduce the same scheme for the case of multivector fields.

\[
[A, B] = \sum_i \frac{\partial A}{\partial \zeta_i} \frac{\partial B}{\partial x_i} - (-1)^{(a-1)(b-1)} \sum_i \frac{\partial B}{\partial \zeta_i} \frac{\partial A}{\partial x_i}
\]

is a \((a + b - 1)\)-vector field.

where

\[
A = \sum_{i_1 < \cdots < i_a} A_{i_1, \ldots, i_a} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_a}} = \sum_{i_1 < \cdots < i_a} A_{i_1, \ldots, i_a} \zeta_{i_1} \cdots \zeta_{i_a}
\]

and

\[
B = \sum_{i_1 < \cdots < i_b} B_{i_1, \ldots, i_b} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_b}} = \sum_{i_1 < \cdots < i_b} B_{i_1, \ldots, i_b} \zeta_{i_1} \cdots \zeta_{i_b}
\]

with

\[
\frac{\partial (\zeta_{i_1} \cdots \zeta_{i_p})}{\partial \zeta_{i_k}} := (-1)^{(p-k)} \eta_{i_1} \cdots \hat{\eta}_{i_k} \eta_{i_{p-1}}
\]
Theorem (Schouten-Nijenhuis)

The bracket defined by this formula satisfies,

**Graded anti-commutativity** \([A, B] = -(-1)^{(a-1)(b-1)} [B, A] \).

**Graded Leibniz rule**

\([A, B \wedge C] = [A, B] \wedge C + (-1)^{(a-1)b} B \wedge [A, C] \)

**Graded Jacobi identity**

\((-1)^{(a-1)(c-1)} [A, [B, C]] + (-1)^{(b-1)(a-1)} [B, [C, A]] + (-1)^{(c-1)(b-1)} [C, [A, B]] = 0 \)

If \(X\) is a vector field then, \([X, B] = L_X B \).
Space of cochains $\mathfrak{x}^m(M)$.

Differential $d_\Pi(A) := [\Pi, A]$.

Poisson cohomology

\[
H^k_\Pi(M) := \frac{\ker d_\Pi : \mathfrak{x}^k(M) \to \mathfrak{x}^{k+1}(M)}{\text{Im} d_\Pi : \mathfrak{x}^{k-1}(M) \to \mathfrak{x}^k(M)}
\]

Computation is difficult. It can be infinite-dimensional. Tools: Mayer-Vietoris, spectral sequences.

Particular cases: $(M, \Pi)$ symplectic $H^k_\Pi(M) \cong H^k_{DR}(M)$.

$(M, \Pi)$ $b$-Poisson, $H^k_\Pi(M) \cong H^k_{DR}(M) \oplus H^{k-1}_{DR}(Z)$. 
Poisson cohomology computation kit

- Hamiltonian vector fields $X_f = -[\Pi, f]$ (1-coboundary).
- Poisson vector fields $[\Pi, X] = -L_X \Pi = 0$ (1-cocycle).
- Poisson structures $[\Pi, \Pi] = 0$ (2-cocycle).
- Compatible Poisson structures $[\Pi_1, \Pi_2] = 0$ (2-cocycle).

\[
H^1_\Pi = \frac{\text{Poisson vector fields}}{\text{Hamiltonian vector fields}}.
\]
Example 5: Cauchy-Riemann equations and Hamilton’s equations

- Take a holomorphic function on $F : \mathbb{C}^2 \to \mathbb{C}$ decompose it as $F = G + iH$ with $G, H : \mathbb{R}^4 \to \mathbb{R}$.

Cauchy-Riemann equations for $F$ in coordinates $z_j = x_j + iy_j$, $j = 1, 2$

$$\frac{\partial G}{\partial x_i} = \frac{\partial H}{\partial y_i}, \quad \frac{\partial G}{\partial y_i} = -\frac{\partial H}{\partial x_i}$$

- Reinterpret these equations as the equality

$$\{G, \cdot\}_0 = \{H, \cdot\}_1 \quad \{H, \cdot\}_0 = -\{G, \cdot\}_1$$

with $\{\cdot, \cdot\}_j$ the Poisson brackets associated to the real and imaginary part of the symplectic form $\omega = dz_1 \wedge dz_2$ ($\omega = \omega_0 + i\omega_1$).

- Check $\{G, H\}_0 = 0$ and $\{H, G\}_1 = 0$ (integrable system).
Example 2: Determinants in $\mathbb{R}^3$ (Exercise 12)

- **Dynamics:** Given two functions $H, K \in C^\infty(\mathbb{R}^3)$. Consider the system of differential equations:
  \[
  (\dot{x}, \dot{y}, \dot{z}) = dH \wedge dK \quad (1)
  \]
  $H$ and $K$ are constants of motion (the flow lies on $H = \text{cte.}$ and $K = \text{cte.}$)

- **Geometry:** Consider the brackets,
  \[
  \{f, g\}_H := \det(df, dg, dH) \quad \{f, g\}_K := \det(df, dg, dK)
  \]
  They are antisymmetric and satisfy Jacobi,
  \[
  \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.
  \]
  The flow of the vector field
  \[
  \{K, \cdot\}_H := \det(dK, \cdot, dH)
  \]
  and $\{-H, \cdot\}_K$ is given by the differential equation (1) and
  \[
  \{H, K\}_H = 0, \quad \{H, K\}_K = 0
  \]
Example 4: Coupling two simple harmonic oscillators

The phase space is \((T^*(\mathbb{R}^2), \omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)\). \(H\) is the sum of potential and kinetic energy,

\[
H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(x_1^2 + x_2^2)
\]

\(H = h\) is a sphere \(S^3\). We have rotational symmetry on this sphere \(\rightsquigarrow\) the angular momentum is a constant of motion, \(L = x_1 y_2 - x_2 y_1\),

\[
X_L = (-x_2, x_1, -y_2, y_1)
\]

and

\[
X_L(H) = \{L, H\} = 0.
\]
An integrable system on a surface.

The invariant submanifolds are tori (Liouville tori)
The orbits of an integrable system in a neighbourhood of a compact orbit are tori. In action-angle coordinates \((p_i, \theta_i)\) the foliation is given by the fibration \(\{p_i = c_i\}\) and the symplectic structure is Darboux
\[
\omega = \sum_{i=1}^{n} dp_i \wedge d\theta_i.
\]
The characters of the day

Joseph Liouville proved the existence of invariant manifolds.

Figure: Joseph Liouville, Henri Mineur, Duistermaat and Arnold

Henri Mineur gave an explicit formula for action coordinates: $p_i = \int_{\gamma_i} \alpha$ where $\gamma_i$ is one of the cycles of the Liouville torus and $\alpha$ is a Liouville 1-form for the symplectic structure ($\omega = d\alpha$).

We will follow the proof by Duistermaat and apply it to Poisson manifolds.
What is an integrable system on a Poisson manifold?

Let \((M, \Pi)\) be a Poisson manifold of (maximal) rank \(2r\) and of dimension \(n\). An \(s\)-tuple of functions \(F = (f_1, \ldots, f_s)\) on \(M\) is said to define a Liouville integrable system on \((M, \Pi)\) if

1. \(f_1, \ldots, f_s\) are independent (\(df_1 \wedge \cdots \wedge df_s \neq 0\)).
2. \(f_1, \ldots, f_s\) are pairwise in involution
3. \(r + s = n\)

Viewed as a map, \(F : M \to \mathbb{R}^s\) is called the moment map of \((M, \Pi)\).
Theorem (Laurent, M., Vanhaecke)

Let $p_1, \ldots, p_r$ be $r$ functions in involution and whose Hamiltonian vector fields are linearly independent at a point $m \in (M, \Pi)$. There exist locally functions $q_1, \ldots, q_r, z_1, \ldots, z_{n-2r}$, such that

1. The $n$ functions $(p_1, q_1, \ldots, p_r, q_r, z_1, \ldots, z_{n-2r})$ form a system of coordinates on $U$, centered at $m$;

2. The Poisson structure $\Pi$ is given on $U$ by

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} + \sum_{i,j=1}^{n-2r} g_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

(1)
Coffee time
An action-angle theorem for Poisson manifolds

Case of regular orbits

We assume that:

1. The mapping $\mathcal{F} = (f_1, \ldots, f_s)$ defines an integrable system on the Poisson manifold $(M, \Pi)$ of dimension $n$ and (maximal) rank $2r$.

2. Suppose that $m \in M$ is a point such that it is regular for the integrable system and the Poisson structure.

3. Assume further than the integral manifold $\mathcal{F}_m$ of the foliation $X_{f_1}, \ldots X_{f_s}$ through $m$ is compact (Liouville torus).
Theorem (Laurent, M., Vanhaecke)

There exist $\mathbb{R}$-valued smooth functions $(p_1, \ldots, p_s)$ and $\mathbb{R}/\mathbb{Z}$-valued smooth functions $(\theta_1, \ldots, \theta_r)$, defined in a neighborhood of $F_m$ such that

1. The functions $(\theta_1, \ldots, \theta_r, p_1, \ldots, p_s)$ define a diffeomorphism $U \simeq T^r \times B^s$;

2. The Poisson structure can be written in terms of these coordinates as

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial \theta_i},$$

in particular the functions $p_{r+1}, \ldots, p_s$ are locally Casimirs of $\Pi$;

3. The leaves of the surjective submersion $\mathcal{F} = (f_1, \ldots, f_s)$ are given by the projection onto the second component $T^r \times B^s$, in particular, the functions $p_1, \ldots, p_s$ depend only on the functions $f_1, \ldots, f_s$. 
Step 1: Topology of the foliation. The fibration in a neighbourhood of a compact connected fiber is a trivial fibration by compact fibers. The fibers are tori.
The Poisson proof

- **Step 2: Hamiltonian action:** We recover a $\mathbb{T}^r$-action tangent to the leaves of the foliation. This implies a process of uniformization of periods.

  \[
  \Phi : \mathbb{R}^r \times (\mathbb{T}^r \times B^s) \rightarrow \mathbb{T}^r \times B^s \\
  ((t_1, \ldots, t_r), m) \mapsto \Phi^{(t_1)} \circ \cdots \circ \Phi^{(t_r)}(m). \tag{2}
  \]

- **Step 3:** We prove that this action is Poisson (if $Y$ is a complete vector field of period 1 and $P$ is a bivector field for which $\mathcal{L}_Y^2 P = 0$, then $\mathcal{L}_Y P = 0$).

- **Step 4:** Finally we use the Poisson Cohomology of the manifold and to check that the action is Hamiltonian.

- **Step 5:** To construct action-angle coordinates we use Darboux-Carathéodory and the constructed Hamiltonian action of $\mathbb{T}^n$ to drag normal forms from a neighbourhood of a point to a neighbourhood of a fiber.
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- **Step 2: Hamiltonian action:** We recover a $\mathbb{T}^r$-action tangent to the leaves of the foliation. This implies a process of uniformization of periods.

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  \Phi : \mathbb{R}^r \times (\mathbb{T}^r \times B^s) \to \mathbb{T}^r \times B^s
  \]

  \[
  ((t_1, \ldots, t_r), m) \mapsto \Phi_{t_1}^{(1)} \circ \cdots \circ \Phi_{t_r}^{(r)}(m).
  \] (2)

- **Step 3:** We prove that this action is Poisson (if $Y$ is a complete vector field of period 1 and $P$ is a bivector field for which $\mathcal{L}_Y^2 P = 0$, then $\mathcal{L}_Y P = 0$).

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  \[ \Phi : \mathbb{R}^r \times (\mathbb{T}^r \times B^s) \rightarrow \mathbb{T}^r \times B^s \]
  \[ ((t_1, \ldots, t_r), m) \mapsto \Phi^{(1)}_{t_1} \circ \cdots \circ \Phi^{(r)}_{t_r}(m). \]  

- **Step 3:** We prove that this action is Poisson (if $Y$ is a complete vector field of period 1 and $P$ is a bivector field for which $L^2_Y P = 0$, then $L_Y P = 0$).

- **Step 4:** Finally we use the Poisson Cohomology of the manifold and to check that the action is Hamiltonian.

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$$((t_1, \ldots, t_r), m) \mapsto \Phi^{(1)}_{t_1} \circ \cdots \circ \Phi^{(r)}_{t_r}(m).$$

Step 3: We prove that this action is Poisson (if $Y$ is a complete vector field of period 1 and $P$ is a bivector field for which $\mathcal{L}_Y^2 P = 0$, then $\mathcal{L}_Y P = 0$).

Step 4: Finally we use the Poisson Cohomology of the manifold and to check that the action is Hamiltonian.

Step 5: To construct action-angle coordinates we use Darboux-Carathéodory and the constructed Hamiltonian action of $\mathbb{T}^n$ to drag normal forms from a neighbourhood of a point to a neighbourhood of a fiber.
**Definition**

Let $(M, \Pi)$ be a Poisson manifold of dimension $n$. An $s$-uplet of functions $\mathcal{F} = (f_1, \ldots, f_s)$ is said to be a **non-commutative integrable system of rank $r$** on $(M, \Pi)$ if

1. $f_1, \ldots, f_s$ are independent;
2. The functions $f_1, \ldots, f_r$ are in involution with the functions $f_1, \ldots, f_s$;
3. $r + s = n$;
4. The Hamiltonian vector fields of the functions $f_1, \ldots, f_r$ are linearly independent at some point of $M$.

Notice that $2r \leq \mathrm{Rk} \Pi$, as a consequence of (4).

**Remark:** The mapping $\mathcal{F} = (f_1, \ldots, f_s)$ is a Poisson map on $\mathbb{R}^s$ with $\mathbb{R}^s$ endowed with a non-vanishing Poisson structure.
An action-angle theorem for non-commutative systems

**Theorem (Laurent, M., Vanhaecke)**

Suppose that $\mathcal{F}_m$ is a regular Liouville torus. Then there exist semilocally $\mathbb{R}$-valued smooth functions $(p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ and $\mathbb{R}/\mathbb{Z}$-valued smooth functions $(\theta_1, \ldots, \theta_r)$ such that,

1. The functions $(\theta_1, \ldots, \theta_r, p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ define a diffeomorphism $U \simeq T^r \times B^s$;

2. The Poisson structure can be written in terms of these coordinates as

\[
\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial \theta_i} + \sum_{k,l=1}^{s-r} \phi_{k,l}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l};
\]

3. The leaves of the surjective submersion $\mathcal{F} = (f_1, \ldots, f_s)$ are given by the projection onto the second component $T^r \times B^s$, in particular, the functions $p_1, \ldots, p_r, z_1, \ldots, z_{s-r}$ depend on the functions $f_1, \ldots, f_s$ only.
The restricted 3-body problem

- Simplified version of the general 3-body problem. One of the bodies has negligible mass.
- The other two bodies move independently of it following Kepler’s laws for the 2-body problem.

**Figure:** Circular 3-body problem
Planar restricted 3-body problem

- The time-dependent self-potential of the small body is 
  \[ U(q, t) = \frac{1-\mu}{|q-q_1|} + \frac{\mu}{|q-q_2|}, \] 
  with \( q_1 = q_1(t) \) the position of the planet with mass \( 1 - \mu \) at time \( t \) and \( q_2 = q_2(t) \) the position of the one with mass \( \mu \).

- The Hamiltonian of the system is 
  \[ H(q, p, t) = \frac{p^2}{2} - U(q, t), \quad (q, p) \in \mathbb{R}^2 \times \mathbb{R}^2, \] 
  where \( p = \dot{q} \) is the momentum of the planet.

- Consider the canonical change \((X, Y, P_X, P_Y) \mapsto (r, \alpha, P_r =: y, P_\alpha =: G)\).

- Introduce **McGehee coordinates** \((x, \alpha, y, G)\), where \( r = \frac{2}{x^2} \), \( x \in \mathbb{R}^+ \), can be then extended to infinity \((x = 0)\).

- The symplectic structure becomes a singular object 
  \[ \omega = -\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG. \] for \( x > 0 \)

- The integrable 2-body problem for \( \mu = 0 \) is integrable with respect to the singular \( \omega \).

\(^1\)Thanks to Amadeu Delshams for this example.
b-Poisson manifolds

Definition (\textit{\textbf{b-integrable system}})

A set of \textit{b}\text{-functions} $f_1, \ldots, f_n$ on $(M^{2n}, \omega)$ such that

- $f_1, \ldots, f_n$ Poisson commute
- $df_1 \wedge \cdots \wedge df_n \neq 0$ as a section of $\Lambda^n (bT^*(M))$ on a dense subset of $M$ and on a dense subset of $\mathbb{Z}$
An action-angle theorem for $b$-Poisson manifolds

**Theorem (Kiesenhofer, M., Scott)**

Let $(M, \Pi)$ be a $b$-Poisson manifold with critical hypersurface $Z$ defined by $t = 0$. Let $f_1, \ldots, f_{n-1}, f_n = \log |t|$ be a $b$-integrable system on it. Then in a neighbourhood of a Liouville torus $m$ there exist coordinates $(\theta_1, \ldots, \theta_n, a_1, \ldots, a_n): U \to T^n \times B^n$ such that

$$\Pi|_U = \sum_{i=1}^{n-1} \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial \theta_i} + ct \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial \theta_n},$$

where the coordinates $a_1, \ldots, a_{n-1}$ depend only on $f_1, \ldots, f_{n-1}$ and $c$ is the modular period of the component of $Z$ containing $m$. 
Perturbation theory KAM on manifolds with boundary and $b$-manifolds. (see Anna’s poster).