An Introduction to Generalized Probabilistic Theories

Peter Janotta

Julius-Maximilians-University Würzburg

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Generalized Probabilistic Theories

- Framework to describe measurement statistics of general physical theories
- Generalization of the density matrix formalism in quantum theory

Motivation

- Why quantum theory? (search for new physical theories)
- Alternatives to quantum theory
Generalized Probabilistic Theories

Usual experimental setup

- Source processing
- State $\omega$
- Measurement $M$ with outcomes $i$
Generalized Probabilistic Theories

Usual experimental setup

Definition (State space)
The state space $\Omega$ is the set of all states $\omega$ allowed in the theory.
Generalized Probabilistic Theories

Usual experimental setup

<table>
<thead>
<tr>
<th>source</th>
<th>processing</th>
<th>measurement M with outcomes i</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td>p₁, p₂, ..., pₘ</td>
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</table>

state ω

Definition (Effects)

Effects are maps \( e^M_i : \Omega \rightarrow [0, 1] \) that give probabilities of measurement outcome \( i \) when measuring state \( ω \):

\[
p(i|ω, M) = e^M_i(ω)
\]

The set of all effects is \( E \)
Probabilistic applications of preparation and measurement devices:

1) Convexity:

\[ \forall \omega = \sum_i \lambda_i \omega_i, \lambda_i \geq 0, \sum_i \lambda_i = 1, \omega_i \in \Omega : \omega \in \Omega \]

\[ \forall e = \sum_i e_i \omega_i, \lambda_i \geq 0, \sum_i \lambda_i = 1, e_i \in E : e \in E \]

\[ \Rightarrow \Omega, E \text{ are convex sets} \]
Minimal physical requirements

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  1) Convexity:

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\[\Rightarrow \Omega, E \text{ are convex sets}\]

Definition (Pure states)

Pure states are given by extremal points in \(\Omega\)
Minimal physical requirements

- Probabilistic applications of preparation and measurement devices:
  2) Linearity (choice of devices should be independent of their measurement statistics):

\[
e(p\omega_1 + (1-p)\omega_2) = pe(\omega_1) + (1-p)e(\omega_2)
\]

\[
[p e_1 + (1-p) e_2](\omega) = p e_1(\omega) + (1-p) e_2(\omega)
\]

⇒ \(e, \omega\) are elements of linear spaces \(A, A^*\)
Minimal physical requirements

- Operationalism: Equivalence of objects leading to the same measurement statistics

\[ e(\omega_1) = e(\omega_2) \forall e \in E \Rightarrow \omega_1 = \omega_2 \]
\[ e_1(\omega) = e_2(\omega) \forall \omega \in \Omega \Rightarrow e_1 = e_2 \]
Minimal physical requirements

- Completeness: all linear mappings $e : \Omega \rightarrow [0, 1]$ included as valid measurement outcomes in a theory

  $\Rightarrow \exists$ unique effect $u : u(\omega) = 1 \quad \forall \omega \in \Omega$ called the unit measure
Minimal physical requirements

- Completeness: all linear mappings \( e : \Omega \to [0, 1] \) included as valid measurement outcomes in a theory

\[ \Rightarrow \exists \text{ unique effect } u : u(\omega) = 1 \quad \forall \omega \in \Omega \text{ called the unit measure} \]

**Definition (Measurement)**

A measurement \( M \) is a set of effects \( \{e_i^M\} \) summing up to the unit measure \( u = \sum_i e_i^M : \)

\[ \Rightarrow \sum_i p(i|\omega, M) = \sum_i e_i^M(\omega) = \left[ \sum_i e_i^M \right](\omega) = u(\omega) = 1 \]
Positive Cones

- State space: vectors on a hyperplane $u(\omega) = 1$
- Unnormalized states form cone $A_+$
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- State space: vectors on a hyperplane \( u(\omega) = 1 \)
- Unnormalized states form cone \( A_+ \)
- Effects: Elements of dual cone \( A^*_+ \)
Examples

Classical Probability Theory

- $\Omega$ is a simplex spanned by $\dim A$ linear independent extremal points $\omega_i$
- For each pure state $\omega_i$ exists an extremal effect $e_i$ that identifies it uniquely: $e_i(\omega_j) = \delta_{ij}$

$$\omega_i = \begin{pmatrix} \cos\left(\frac{2\pi(i-1)}{4}\right) \\ \sin\left(\frac{2\pi(i-1)}{4}\right) \\ 1 \end{pmatrix}$$

$$e_i = \omega_i$$

$$u = (1, 1, 1)^T$$

- $\omega_i$ form basis for $A \Rightarrow$ unique decomposition of mixed states only for classical systems
Examples

Quantum theory

- \( A_+ = A_+^* \) is the cone of positive hermitian matrices \( \rho \geq 0, \rho^\dagger = \rho \)
- \( e(\rho) = \text{tr}[e.\rho], \quad u = 1 \)
\( \Rightarrow \Omega = \{\rho \in A_+ | u(\rho) = \text{tr}[\rho] = 1\} \)

Qubit: \( \Omega \) is given by 3-dim Bloch sphere
The Gbit

- $\Omega$ given by a square

- No uncertainty for pure states with respect to measurements $M_1 = \{e_1, e_3\}$ and $M_2 = \{e_2, e_4\}$
Joint Systems

Definition (Global State Assumption)

Joint system $AB$ fully characterized by joint probabilities $\{p(e^A, e^B)\}$
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**Definition (No-Signalling Principle)**
Local operations do not change the measurement statistic in other parts of the system
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Global State Assumption
No-Signalling Principle
\[ \implies AB_+ \subseteq A \otimes B \]
Cone Of Joint Systems

- Cone $AB_+$ of joint system bounded:

$$A_+ \otimes_{\text{min}} B_+ \subseteq AB_+ \subseteq A_+ \otimes_{\text{max}} B_+$$
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- Maximal tensor product (separable + entangled):
  \[ A_+ \otimes_{\max} B_+ = \{\omega^{AB} \in A \otimes B \mid \omega^{AB}(e^A \otimes e^B) \geq 0\} \]
  includes all joint states allowed by No-Signalling
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- Theorie defined by structure of $A_+, u^A, B_+, u^B$ and $AB_+ ...$
### Duality relations

For finite dimensional systems:

\[
(A_+ \otimes_{\text{max}} B_+) = (A_+^* \otimes_{\text{min}} B_+^*)^*
\]

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- All tensor products equal or there exists at least either states or effects that are entangled
- In QT: Balanced between entangled effects and states
Thank you for your attention!