# An Introduction to Generalized Probabilistic Theories

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# Generalized Probabilistic Theories

## Generalized Probabilistic Theories

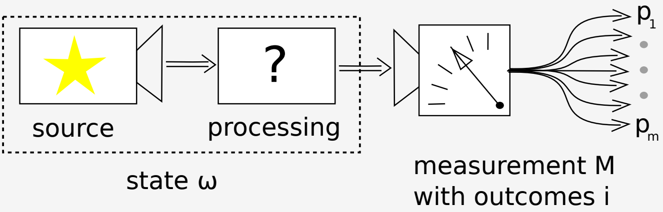
- Framework to describe measurement statistics of general physical theories
- Generalization of the density matrix formalism in quantum theory

## Motivation

- Why quantum theory? (search for new physical theories)
- Alternatives to quantum theory

# Generalized Probabilistic Theories

Usual experimental setup



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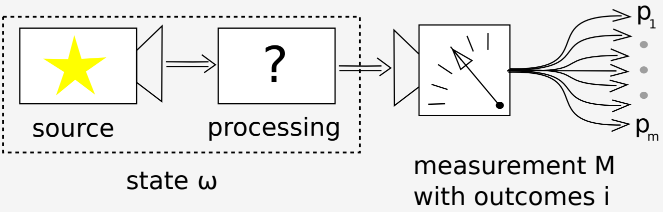


## Definition (State space)

The state space $\Omega$ is the set of all states $\omega$ allowed in the theory

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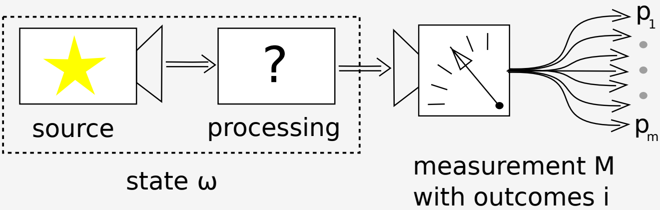


## Definition (Effects)

Effects are maps $e_i^M : \Omega \to [0,1]$ that give probabilities of measurement outcome $i$ when measuring state $\omega$:

$$p(i|\omega, M) = e_i^M(\omega)$$

The set of all effects is $E$

# Minimal physical requirements

- Probabilistic applications of preparation and measurement devices:
  1) Convexity:

$$\forall \omega = \sum_i \lambda_i \, \omega_i, \lambda_i \geq 0, \sum_i \lambda_i = 1, \omega_i \in \Omega : \omega \in \Omega$$

$$\forall e = \sum_i e_i \, \omega_i, \lambda_i \geq 0, \sum_i \lambda_i = 1, e_i \in E : e \in E$$

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$\Rightarrow \Omega, E$ are convex sets

## Definition (Pure states)

Pure states are given by extremal points in $\Omega$

# Minimal physical requirements

- Probabilistic applications of preparation and measurement devices:

  2) Linearity (choice of devices should be independent of their measurment statistics):

$$e\,(p\,\omega_1 + (1-p)\,\omega_2) = p\,e(\omega_1) + (1-p)\,e(\omega_2)$$
$$[p\,e_1 + (1-p)\,e_2]\,(\omega) = p\,e_1(\omega) + (1-p)\,e_2(\omega)$$

$\Rightarrow e, \omega$ are elements of linear spaces $A, A^*$

# Minimal physical requirements

- Operationalism: Equivalence of objects leading to the same measurement statistics

$$e(\omega_1) = e(\omega_2) \forall e \in E \Rightarrow \omega_1 = \omega_2$$
$$e_1(\omega) = e_2(\omega) \forall \omega \in \Omega \Rightarrow e_1 = e_2$$

# Minimal physical requirements

- Completeness: all linear mappings $e : \Omega \to [0, 1]$ included as valid measurement outcomes in a theory

  $\Rightarrow \exists$ unique effect $u : u(\omega) = 1 \quad \forall \omega \in \Omega$ called the unit measure

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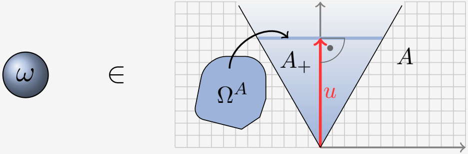
## Definition (Measurement)

A measurement $M$ is a set of effects $\{e_i^M\}$ summing up to the unit measure $u = \sum_i e_i^M$:

$$\Rightarrow \sum_i p(i|\omega, M) = \sum_i e_i^M(\omega) = \left[\sum_i e_i^M\right](\omega) = u(\omega) = 1$$

# Positive Cones

- State space: vectors on a hyperplane $u(\omega) = 1$
- Unnormalized states form cone $A_+$



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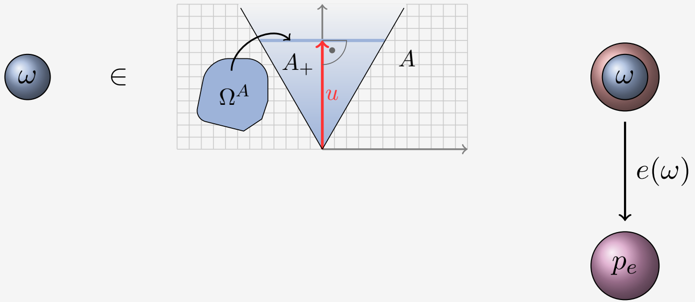

# Positive Cones

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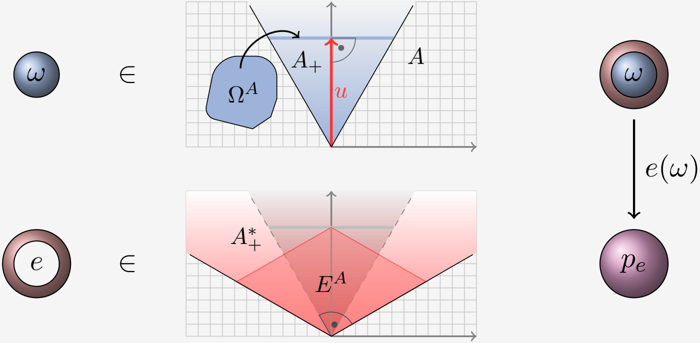
- Unnormalized states form cone $A_+$
- Effects: Elements of dual cone $A_+^*$



# Examples

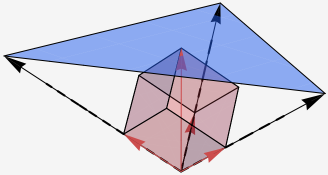
Classical Probability Theory

- $\Omega$ is a simplex spanned by $dim^A$ linear independent extremal points $\omega_i$
- For each pure state $\omega_i$ exists an extremal effect $e_i$ that identifies it uniquely: $e_i(\omega_j) = \delta_{ij}$

$$\omega_i = \begin{pmatrix} cos(\frac{2\pi(i-1)}{4}) \\ sin(\frac{2\pi(i-1)}{4}) \\ 1 \end{pmatrix}$$

$$e_i = \omega_i$$

$$u = (1,1,1)^T$$



n=3

- $\omega_i$ form basis for $A \Rightarrow$ unique decomposition of mixed states only for classical systems

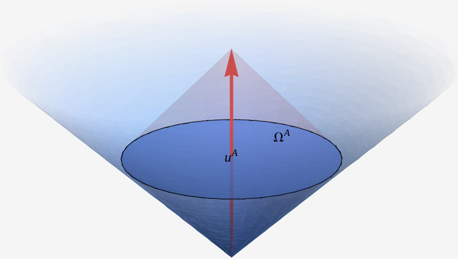
# Examples

Quantum theory

- $A_+ = A_+^*$ is the cone of positive hermitian matrices $\rho \geq 0, \rho^\dagger = \rho$
- $e(\rho) = \mathrm{tr}[e.\rho] \quad u = \mathbb{1}$

$\Rightarrow \Omega = \{\rho \in A_+ | u(\rho) = \mathrm{tr}[\rho] = 1\}$

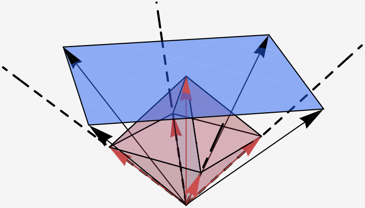
Qubit: $\Omega$ is given by 3-dim Bloch sphere



# Examples

The Gbit

- $\Omega$ given by a square




- No uncertainty for pure states with respect to measurements $M_1 = \{e_1, e_3\}$ and $M_2 = \{e_2, e_4\}$

# Joint Systems

## Definition (Global State Assumption)

Joint system $AB$ fully characterized by joint probabilities $\{p(e^A, e^B)\}$

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$$\left.\begin{array}{l}\text{Global State Assumption}\\ \text{No-Signalling Principle}\end{array}\right\} \Longrightarrow \quad AB_+ \quad \subseteq \quad A \otimes B$$

# Cone Of Joint Systems

- Cone $AB_+$ of joint system bounded:

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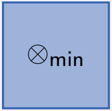
$$A_+ \otimes_{\text{min}} B_+ = \text{ConvexSpan}\{\omega^A \otimes \omega^B\}$$



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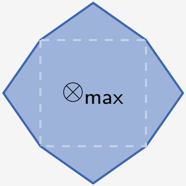


- Maximal tensor product (separable + entangled):

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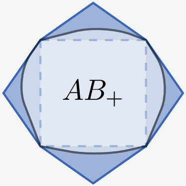


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- Theorie defined by structure of $A_+, u^A$, $B_+, u^B$ and $AB_+$ ...

# Duality relations

For finite dimensional systems:

$$(A_+ \otimes_{\max} B_+) = \left(A_+^* \otimes_{\min} B_+^*\right)^*$$
$$(A_+ \otimes_{\min} B_+) = \left(A_+^* \otimes_{\max} B_+^*\right)^*$$

- All tensor products equal or there exists at least either states or effects that are entangled
- In QT: Balanced between entangled effects and states

# Thank you for your attention!