Infinite dimensional quantum systems from the finite dimensional viewpoint

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Objective of the talk

Most of the quantum systems encountered in practice are governed by PDEs

$$\mathrm{i}\frac{\partial\psi}{\partial t}(x,t) = (-\Delta + V(x))\psi(x,t) + u(t)W(x)\psi(x,t)$$

We will try to understand how the properties (controllability) of these infinite dimensional systems can be deduced from the properties of their finite dimensional approximations.

In what follows, we neglect decoherence.

Outline of the talk

1. Finite dimensional bilinear quantum systems
   - Bilinear systems in compact groups
   - Controllability
   - Control in practice
2. Infinite dimensional quantum systems
   - Bilinear Schrödinger equation
   - Obstructions to controllability
   - Controllability results
3. Finite dimensional viewpoint
   - Good Galerkin approximation
   - Rotation of a planar molecule
   - Quantum harmonic oscillator

Outline of the talk

Bilinear control systems

Let A and B be two $n \times n$ skew hermitian matrices ($\overline{A}^T = -A$, $\overline{B}^T = -B$), and fix x_0 in $\mathbf{C}^n$. For scalar valued u, we consider

$$(\sigma) \left\{ \begin{array}{ll} x'(t) & = (A + u(t)B)x(t) \\ x(0) & = x_0 \end{array} \right.$$

Proposition

For every $u : \mathbf{R} \to \mathbf{R}$, for every x_0 in $\mathbf{C}^n$, the solution $t \mapsto X_t^u x_0$ of (σ) lies, for every time, in the Hilbert sphere of $\mathbf{C}^n$ containing x_0.

Lift in matrix groups

Sussmann, 70's

This system in $\mathbf{C}^n$

$$(\sigma)\left\{\begin{array}{ll} x'(t) & = (A+u(t)B)x(t) \\ x(0) & = x_0 \end{array}\right.$$

can be lift in $U(n)$

$$(\Sigma)\left\{\begin{array}{ll} X'(t) & = (A+u(t)B)X(t) \\ X(0) & = \mathrm{Id}_n \end{array}\right.$$

Proposition

For every $u:\mathbf{R}\to\mathbf{R}$, the solution X_t^u of (Σ) lies, for every time t, in $U(n)=\{M\in\mathcal{M}_{n,n}(\mathbf{C}) | \overline{M}^T M = \mathrm{Id}_n\}$.

Notions of controllability

Definition

The control system σ is controllable on the unit sphere of $\mathbf{C}^n$ if for every x_0, x_1, there exists $u : [0, T] \to \mathbf{R}$ such that $X_T^u x_0 = x_1$.

Definition

The control system Σ is controllable in $U(n)$ if for every X_1 in $U(n)$, there exists $u : [0, T] \to \mathbf{R}$ such that $X_T^u = X_1$.

Attainable set

A, B smooth vector fields on the manifold M

$$\dot{x} = A(x) + u(t)B(x), \quad x \in M$$

Definition

Solution at time t with control u from x_0: $\Upsilon_t^u(x_0)$.
Attainable set at time t $\mathcal{A}_t(x_0) = \{\Upsilon_t^u(x_0) : u \in L^1([0,t])\}$
Attainable set $\mathcal{A}(x_0) = \cup_{t \geq 0} \mathcal{A}_t(x_0)$

Accessibility

$$\dot{x} = A(x) + u(t)B(x), \quad x \in M$$

Definition (Lie bracket)

$$[A,B](x) = \frac{dA}{dx}B - \frac{dB}{dx}A$$

The Lie algebra $\mathrm{Lie}(A,B)$ spanned by A and B is the linear subspace of $Vec(M)$ spanned by all the brackets, of any length, of A and B ($[A,B]$, $[A,[A,B]]$, $[B,[A,B]]$, $\ldots$).

Proposition (Krener's theorem, Jurdjevic-Sussmann, 1973)

If $\mathrm{Lie}_x(A,B) = T_xM$, then $\mathcal{A}(x)$ is contained in the closure of its interior (i.e., is not an hairy set).

A criterion for controllability in $U(n)$

$$\dot{x} = Ax + u(t)Bx, \qquad M = U(n)$$

$$[A, B] = AB - BA$$

Proposition (Jurdjevic-Sussman, 1972)

(Σ) *is controllable in* $U(n)$ *if and only if* $\mathrm{Lie}(A, B) = \mathfrak{u}(n)$.

- Fundamental theoretical result.
- Use with caution in practice ($\dim U(n) = n^2 - 1$, how many brackets do you have to compute?)

Choice of a basis

Up to a conjugation, one may assume that A is diagonal.

$$A = \begin{pmatrix} i\lambda_1 & 0 & \cdots & 0 \\ 0 & i\lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & i\lambda_n \end{pmatrix} B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{n,n-1} & b_{nn} \end{pmatrix}$$

Non resonant transitions

Definition

A transition (j,k), $j \neq k$, is non resonant if $b_{j,k} \neq 0$ and, for every l_1, l_2,

$$|\lambda_{l_1} - \lambda_{l_2}| = |\lambda_j - \lambda_k| \implies \{l_1, l_2\} = \{j, k\} \text{ or } b_{l_1, l_2} = 0.$$

Definition

A transition (j,k), $j \neq k$, is strongly non resonant if $b_{j,k} \neq 0$ and, for every l_1, l_2,

$$\frac{|\lambda_{l_1} - \lambda_{l_2}|}{|\lambda_j - \lambda_k|} \in \mathbf{Z} \implies \{l_1, l_2\} = \{j, k\} \text{ or } b_{l_1, l_2} = 0.$$

Lyapounov techniques

Define the distance to the target $V(\psi) = \|\psi - \psi_{ref}\|^2$. Consider a supplementary control ω

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = (A + \mathrm{i}\omega)\psi + u(t)B\psi$$

At every time t, chose $u(t) = \Re\langle B\psi, \psi_{ref}\rangle$ and $\omega(t) = \lambda + \Re\langle\psi, \psi_{ref}\rangle$ [such that $\frac{d}{dt}V(\psi(t)) < 0$].

Proposition (Mirrahimi-Rouchon-Turinici, 2005)

For allmost every A and B, if ψ_{ref} is an eigenstate of A, then for almost every λ in $\mathbf{R}$, the trajectory converges to the target

Periodic control laws (strongly non resonant case)

Proposition

Let (j,k) be a strongly non resonant transition and u^* be a $\frac{2\pi}{|\lambda_j-\lambda_k|}$-periodic function. If $\int_0^{\frac{2\pi}{|\lambda_j-\lambda_k|}} u^*(\tau)e^{\mathrm{i}|\lambda_j-\lambda_k|\tau}\mathrm{d}\tau \neq 0$, then there exists T^* such that

$$\left|\langle \phi_k, X^{u^*/n}_{nT^*}\phi_j\rangle\right| \overset{n\to\infty}{\longrightarrow} 1.$$

$$T^* = \frac{\pi T}{2|b_{j,k}|\left|\int_0^T u^*(\tau)e^{\mathrm{i}(\lambda_j-\lambda_k)\tau}\mathrm{d}\tau\right|}.$$

Some estimates

L^1 norm needed to achieve the transition from level j to k

$$\frac{\pi}{2|b_{jk}|}\frac{1}{\mathrm{Eff}_{jk}(u^*)}$$

with

$$0 \leq \mathrm{Eff}_{jk}(u^*) = \frac{\left|\int_0^{\frac{2\pi}{|\lambda_j-\lambda_k|}} u^*(\tau)e^{\mathrm{i}(\lambda_j-\lambda_k)\tau}\mathrm{d}\tau\right|}{\int_0^{\frac{2\pi}{|\lambda_j-\lambda_k|}} |u^*(\tau)|\mathrm{d}\tau} \leq 1.$$

Error estimates

$$1 - |\langle \phi_k, X^{u^*/n}_{nT^*}\phi_j\rangle| \leq \frac{C(u^*,B)}{n \inf_{l_1,l_2,l_3}\left|\frac{|\lambda_{l_1}-\lambda_{l_2}|}{|\lambda_j-\lambda_k|} - l_3\right|}$$

$$\mathrm{Error} \times \mathrm{Time} \leq \mathrm{Const}$$

Periodic control laws (general non resonant case)

Proposition

Let (j,k) be a non resonant transition and u^* be a $\frac{2\pi}{|\lambda_j-\lambda_k|}$-periodic function. If $\mathrm{Eff}_{jk}(u^*)\neq 0$ and $\mathrm{Eff}_{l_1 l_2}(u^*)=0$ for every l_1,l_2 such that $\frac{|\lambda_{l_1}-\lambda_{l_2}|}{|\lambda_j-\lambda_k|}\in\mathbf{Z}$ and $\{l_1,l_2\}\neq\{j,k\}$, then there exists T^* such that

$$\left|\langle\phi_k, X^{u^*/n}_{nT^*}\phi_j\rangle\right| \overset{n\to\infty}{\longrightarrow} 1.$$

L^1 norm estimates:

$$\|u\|_{L^1}\leq\frac{\pi}{2\mathrm{Eff}_{jk}(u^*)|b_{jk}|}$$

One may chose u^* such that

$$\mathrm{Eff}_{jk}=\prod_{l=2}^{\infty}\cos\left(\frac{\pi}{2k}\right)\approx 0.43$$

Numerical simulations (strongly non resonant case

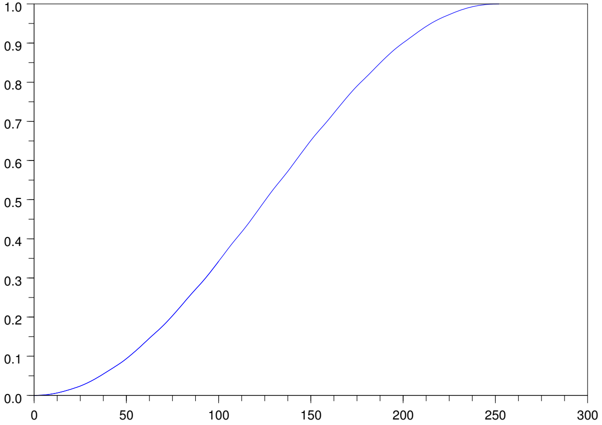
$$A = -\mathrm{i}\ \mathrm{diag}(1^2, 2^2, 3^2, \ldots, N^2)$$

$$B = -\mathrm{i}\begin{pmatrix} 0 & 1/2 & 0 & \cdots & \cdots & 0 \\ 1/2 & 0 & 1/2 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1/2 \\ 0 & \cdots & \cdots & 0 & 1/2 & 0 \end{pmatrix}$$

We chose $N = 22$.

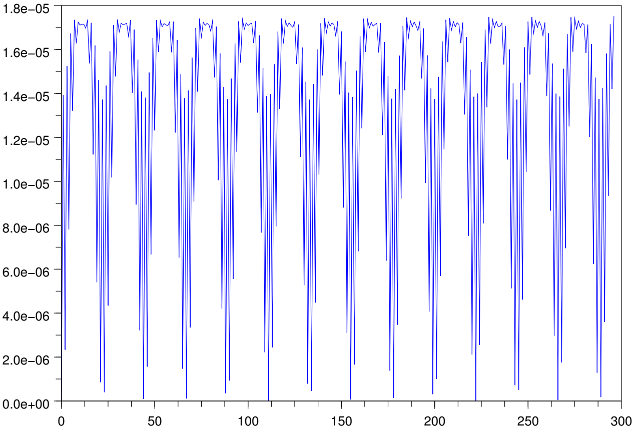
Numerical simulations: strongly non resonant case

$u^*(t) = \cos^3(t)$, $\mathrm{Eff}_{1,2}(u^*) = 9\pi/32 \approx 0.88$, $n = 30$



Periodic control laws: numerical simulations

$u^*(t) = \cos^2(t)$, $\mathrm{Eff}_{1,2}(u^*) = 0$, $n = 30$



Numerical simulations (general non resonant case)

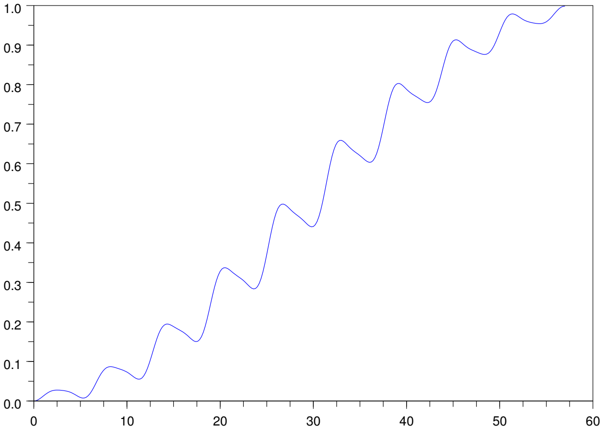
$$A = -\mathrm{i}\ \mathrm{diag}(0^2, 1^2, 2^2, 3^2, \ldots, N^2)$$

$$B = -\mathrm{i} \begin{pmatrix} 0 & \sqrt{2}/2 & 0 & \cdots & \cdots & 0 \\ \sqrt{2}/2 & 0 & 1/2 & 0 & & \vdots \\ 0 & 1/2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1/2 & 0 & 1/2 \\ 0 & \cdots & \cdots & 0 & 1/2 & 0 \end{pmatrix}$$

We chose $N = 22$.

Numerical simulations: general non resonant case

$u^*(t) = 3\cos(t)/2 + 2,\ \mathrm{Eff}_{1,2}(u^*) = 3/8,\ \mathrm{Eff}_{2,3}(u^*) = 0,\ n = 20$



Outline of the talk

Some examples

A quantum system evolving in Ω, a finite dimensional Riemannian manifold, is described by its *wave function* ψ in the unit sphere of $L^2(\Omega, \mathbf{C})$. The system is in the subset ω with probability $\int_\omega |\psi|^2 \mathrm{d}\mu$. The time evolution is given by the Schrödinger equation

$$\mathrm{i}\frac{\partial \psi}{\partial t}(x,t) = (-\Delta + V(x))\psi(x,t)$$

When submitted to an external field (*e.g.*, a laser) with time variable intensity, ψ satisfies

$$\mathrm{i}\frac{\partial \psi}{\partial t} = (-\Delta + V(x))\psi(x,t) + u(t)W(x)\psi(x,t)$$

Some examples

Rotation of a planar molecule

$\Omega = SO(2) \simeq \mathbf{R}/2\pi\mathbf{Z}$

$$\mathrm{i}\frac{\partial \psi}{\partial t}(\theta, t) = -\partial_{\theta\theta}\psi(\theta, t) + u(t)\cos\theta\psi(\theta, t)$$

Rotation of a molecule in space

$\Omega = S^2$

$$\mathrm{i}\frac{\partial \psi}{\partial t}(\theta, \nu, t) = -\Delta\psi(\theta, \nu, t) + u(t)\cos\theta\psi(\theta, \nu, t)$$

Some examples

Harmonic oscillator

$\Omega = \mathbf{R}$

$$\mathrm{i}\frac{\partial \psi}{\partial t}(x,t) = (-\partial_{xx} + x^2)\psi(x,t) + u(t)x\psi(x,t)$$

Infinite square potential well

$\Omega = (0,\pi)$

$$\mathrm{i}\frac{\partial \psi}{\partial t}(x,t) = \partial_{x,x}\psi(x,t) + u(t)x\psi(x,t)$$

Abstract form

In the Hilbert space $H(=L^2(\Omega,\mathbf{C})$, we consider an unbounded skew-adjoint linear operator $A(=-\mathrm{i}(\Delta+V))$, a skew symmetric operator $B(=-\mathrm{i}W(x))$ and the evolution equation

$$\frac{d\psi}{dt}=(A+u(t)B)\psi$$

Well-posedness

$$\begin{cases} \frac{d\psi}{dt} = (A + u(t)B)\psi \\ \psi(0) = \psi_0 \end{cases}$$

Well-posedness is very far from obvious. Cauchy-Lipschitz Theorem does not apply when A is unbounded (i.e., not continuous), what is the case here.

In the presented examples, for every locally integrable $u : \mathbf{R} \to \mathbf{R}$, we can define the solution $t \mapsto \Upsilon^u_t(\psi_0)$. If ψ_0 belongs to $D(A)$, then $\Upsilon^u(\psi_0)$ is absolutely continuous and

$$\frac{d}{dt}\Upsilon^u_t(\psi_0) = (A + u(t)B)\Upsilon^u_t(\psi_0) \quad \text{for a.e.} t$$

Discrete spectrum

In the presented examples, A has discrete spectrum. There exists a non-decreasing sequence $(\lambda_n)_{n\in\mathbf{N}}$ in $[0,+\infty)$ and an Hilbert basis $(\psi_n)_{n\in\mathbf{N}}$ of H such that $A\psi_n = -\mathrm{i}\lambda_n\phi_n$ for every n.
Infinite dimensional matrices representation

$$A = \begin{pmatrix} -\mathrm{i}\lambda_1 & 0 & \cdots & \cdots \\ 0 & -\mathrm{i}\lambda_2 & \ddots & \\ \vdots & \ddots & -\mathrm{i}\lambda_3 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

$$b_{j,k} = \langle \phi_j, B\phi_k \rangle, \qquad b_{jk} = -\overline{b_{kj}}$$

Ball Marsden Slemrod

Theorem (Ball Marsden Slemrod, 1981 and Turinici, 2000)

If B is bounded, then the attainable set has empty interior in the intersection of $D(A)$ with the unit sphere of H.

Briefly: exact controllability is hopeless. It does not prevent approximate controllability (or exact controllability on a smaller set).

Regularity issues

Proposition

If $D(A^k)$ is invariant for the unitary transformations $e^{(A+uB)}$, $u \in \mathbf{R}$, then $D(A^k)$ is stable for the dynamics of the system (also for non constant controls u).

- This is a case, for every k, for all the examples encountered in the litterature but the infinite square potential well.
- The eigenstates belong the $D(A^k)$ for every k.

Connectedness issues

If the matrix of B in the basis $(\phi_n)_{n\in\mathbf{N}}$ is not connected, no global controllability (in any sense) is to be expected.
Example: rotation of a planar molecule;

$$\mathrm{i}\frac{\partial\psi}{\partial t}\psi(\theta,t) = -\partial_{\theta\theta}\psi(\theta,t) + u(t)\cos\theta\psi(\theta,t)$$

$\cos\theta$ does not couple odd eigenfunctions with even ones.

$$A = -\mathrm{i}\ \mathrm{diag}(0, 1^2, 1^2, 2^2, 2^2, 3^2, 3^2, \ldots)$$

Each eigenvalue but 0 is double and associated with two orthogonal eigenfunctions ϕ_j^e and ϕ_j^o.

$$\langle\phi_j^e, B\phi_k^o\rangle = 0 \text{ for every } \{j,k\}$$

Quantum harmonic oscillator

Theorem (Mirrahimi-Rouchon,2004)

The quantum harmonic oscillator is not controllable, in any reasonable sense.

$$A = -\mathrm{i}\ \mathrm{diag}(1/2, 3/2, 5/2, \ldots)$$

$$B = -\mathrm{i}\begin{pmatrix} 0 & \sqrt{1} & 0 & \ldots \\ \sqrt{1} & 0 & \sqrt{2} & \ddots \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ \vdots & \ddots & \sqrt{3} & \ddots \end{pmatrix}$$

All the Galerkin approximations are exactly controllable.

Square potential well

$\Omega = (0,1)$

$$\frac{\partial \psi}{\partial t}(x,t) = \underbrace{\mathrm{i}\Delta\psi(x,t)}_{A\psi} + u(t)\underbrace{W(x)\psi(x,t)}_{B\psi}$$

Theorem (Beauchard-Laurent, 2009)

If there exists $C > 0$ such that for every $j \in \mathbf{N}$,

$$|b_{1,j}| > \frac{C}{j^3}$$

then the system is exactly controllable in the intersection of the unit sphere with $H^3_{(0)}$.

Square potential well

The proof relies on moment theory (to prove surjectivity of the differential of the input-output mapping) and a fixed point theorem (in an infinite dimensional Banach space).

- Very precise result.
- By far, the best available result on the structure of the attainable set of a bilinear quantum control system.
- Not constructive.
- Irregular controls (L^2 controls).
- Extension to examples in dimension greater than one is an open question (very hard).

Weyl's estimate for the k^{th} eigenvalue of the Laplacian on a d-dimensional compact manifold:

$$\lambda_k \sim C k^{\frac{d}{2}}$$

Lyapunov techniques

$$i\frac{\partial \psi}{\partial t}(x,t) = \underbrace{-\Delta\psi(x,t) + V(x)\psi(x,t)}_{A\psi} + u(t)\underbrace{W(x)\psi(x,t)}_{B\psi}$$

Ω is a bounded domain of $\mathbf{R}^{\mathbf{d}}$, with smooth boundary.

Theorem (Nersesyan, 2009)

If

- $b_{1,j} \neq 0$ *for every* $j \geq 1$ *and*
- $|\lambda_1 - \lambda_j| \neq |\lambda_k - \lambda_l|$ *for every* $j > 1$, $\{1, j\} \neq \{k, l\}$

then the control system is approximately controllable on the unit sphere for H^s *norms.*

Geometric techniques

Hypotheses:

- A is skew adjoint with discrete spectrum $(-\mathrm{i}\lambda_n)_{n\in\mathbf{N}}$
- for every u in $\mathbf{R}^+$, $A+uB$ is essentially skew adjoint

Geometric techniques

Definition

A subset S of $\mathbf{N}^2$ connects the levels j and k if there exists a finite sequence $j = l_0, l_1, \ldots, l_p = k$ such that (l_m, l_{m+1}) belongs to S for $m < p$ and $\langle \phi_{l_m}, B\phi_{l_{m+1}} \rangle \neq 0$.

Definition

A connected chain is a set that links every pair of integers. A connected chain S is said to be non resonant if for every (l_1, l_2) in S, j, j' in $\mathbf{N}^2$, $|\lambda_{l_1} - \lambda_{l_2}| = |\lambda_j - \lambda_{j'}|$ implies $\{l_1, l_2\} = \{j, j'\}$ or $\langle \phi_j, B\phi_{j'} \rangle = 0$.

Geometric techniques

Theorem (Boscain-Chambrion-Caponigro-Sigalotti, 2011)

If (A, B) admits a non resonant chain of connectedness S, then, for every $\delta > 0$, (A, B) is approximately simultaneously controllable by means of piecewise constant functions taking value in $(0, \delta)$.

If (j, k) belongs to S, then the L^1 norm needed to join (approximately) j and k is less than

$$\frac{\pi}{2\nu|\langle \phi_j, B\phi_k \rangle|}, \text{ with } \nu = \prod_{l \geq 2} \cos\left(\frac{\pi}{2l}\right) \approx 0.43.$$

Geometric techniques

Some nice points:

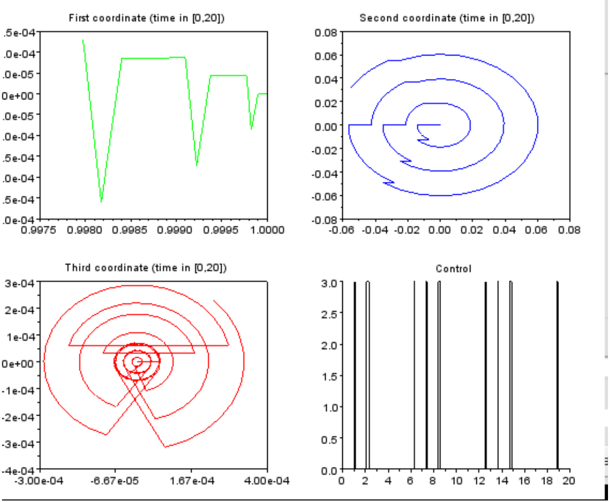
- Very general result;
- No hypotheses on the regularity on A or B (applies to very wild situations);
- "Constructive" proof;
- Provides very precise estimates on the L^1-norm of the control.

Some major drawbacks

- Very weak result: it does not say anything about the structure of the attainable set.
- No estimate for the time.
- No estimate for the error.

Numerical simulations

Rotation of a planar molecule (odd subspace).



Outline of the talk

Weakly coupled quantum systems

Assumptions:

- A is skew-adjoint with discrete spectrum $(-\mathrm{i}\lambda_n)_{n\in\mathbf{N}}$, $(\lambda_n)_{n\in\mathbf{N}}$ is non decreasing and tends to infinity.
- B is bounded, skew-adjoint.
- for every u in $\mathbf{R}$, $D(A+uB)=D(A)$ and $D((A+uB)^2)=D(A^2)$.

Definition (Weakly coupled system)

(A,B) is weakly-coupled if there exists $C_{A,B}$ such that, for every ψ in $D(A)$,

$$|\Im\langle A\psi, B\psi\rangle| \leq C_{A,B}|\langle A\psi,\psi\rangle|$$

Growth of $|A|^{1/2}$ norms

$|\langle A\psi,\psi\rangle| = \sum_{n\in\mathbf{N}} \lambda_n |\langle \phi_n,\psi\rangle|$ is the expected value of the energy at ψ.

$$\begin{aligned}\left|\frac{d}{dt}\langle |A|\psi,\psi\rangle\right| &= 2\Re\langle |A|\psi,(A+u(t)B)\psi\rangle \\ &\leq 2|u(t)|C_{A,B}|\langle A\psi,\psi\rangle|\end{aligned}$$

By Gronwall's lemma:

$$|\langle A\psi(t),\psi(t)\rangle| \leq e^{2C_{\boldsymbol{A},\boldsymbol{B}}\int_{\mathbf{0}}^{\boldsymbol{t}}|u(\tau)|\mathrm{d}\tau}|\langle A\psi(0),\psi(0)\rangle|$$

Size of velocity tail

Define $\pi_N : H \to H$, the orthogonal projection on the first N eigenstates of A.

$$\begin{aligned}
\|B(\mathrm{Id}-\pi_N)\psi(t)\|^2 &\leq \|B\|^2 \sum_{n\geq N} |\langle \phi_n, (\mathrm{Id}-\pi_N)\psi(t), \rangle|^2 \\
&\leq \frac{1}{\lambda_N}\|B\|^2 \sum_{n\geq N} \lambda_n |\langle \phi_n, (\mathrm{Id}-\pi_N)\psi(t), \rangle|^2 \\
&\leq \frac{1}{\lambda_N}\|B^2\|^2 |\langle A(\mathrm{Id}-\pi_N)\psi(t), (\mathrm{Id}-\pi_N)\psi(t)\rangle| \\
&\leq \frac{\|B^2\|^2 e^{2C_{A,B}\|u\|_{L^1}} |\langle A\psi(0), \psi(0)\rangle|}{\lambda_N} \to 0
\end{aligned}$$

Good Galerkin approximation

$$\pi_N \psi'(t) = A^{(N)} \pi_N \psi(t) + u(t)\pi_N B \pi_N \psi(t) + u(t)\pi_N B(1-\pi_N)\psi(t)$$

Denoting with $X_u^{(N)}(t)$ the propagator of the N-dimensional system $x' = (A^{(N)} + u(t)B^{(N)})x$,

$$\pi_N \psi(t) = X_u^{(N)}(t)\pi_N \psi(0) + \int_0^t X_u^{(N)}(t,s)u(\tau)\pi_N B(1-\pi_N)\psi(\tau)\mathrm{d}\tau$$

Proposition (Boussaid-Caponigro-Chambrion, 2011)

Let (A,B) be weakly-coupled. For every $\epsilon > 0$, for every $K > 0$, for every ψ_0, there exists $N = N(\epsilon, K, \psi_0)$ such that

$$\|u\|_{L^1} \leq K \implies \|\pi_N \Upsilon_t^u(\psi_0) - X_u^{(N)}(t)\pi_N \psi_0\| < \epsilon.$$

Good Galerkin approximation

- Many estimates are known for the A^k norms of the solutions of the Schrödinger equation (see Bourgain, 1999). The point is that this estimate is uniform with respect to the control.
- The result also applies to more general cases (B unbounded, A with non-discrete spectrum) but requires regularity.
- Possible extensions to A with continuous spectrum (WIP).

Good Galerkin approximation

We restrict to the odd subspace.

$$A = -\mathrm{i}\ \mathrm{diag}(1^2, 2^2, 3^2, \dots, N^2)$$

$$B = -\mathrm{i} \begin{pmatrix} 0 & 1/2 & 0 & \cdots & 0 & \\ 1/2 & 0 & 1/2 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & & \\ 0 & \dots & \dots & 0 & 1/2 & 0 \end{pmatrix}$$

For $\psi_0 = \phi_1$, $\epsilon = 10^{-3}$ and $K = 14/3$, one finds $N = 22$.

Controllability

- What we did provides a constructive proof of the controllability of the 2D planar molecule.
- When the cost is the L^1 norm, a minimizing sequence of controls is given by periodic Dirac functions. [The corresponding efficiencies tend to 1.]
- Unknown form of a time minimizing sequence of controls.

Quantum harmonic oscillator: Good Galerkin approximation

$$A = -\mathrm{i}\ \mathrm{diag}(1/2, 3/2, 5/2, \ldots)$$

$$B = -\mathrm{i} \begin{pmatrix} 0 & \sqrt{1} & 0 & \ldots \\ \sqrt{1} & 0 & \sqrt{2} & \ddots \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ \vdots & \ddots & \sqrt{3} & \ddots \end{pmatrix}$$

B is not bounded. However, B is bounded relatively to A and the system still admits a sequence of Good Galerkin approximations. $\epsilon = 10^{-3}$, $K = 3$, $N \approx 400$

Controllability of the infinite dimensional system?

Scheme of the proof:

1. Find a sequence of Galerkin approximations that are controllable.
2. Prove that these Galerkin approximations are controllable with a uniformly bounded L^1-norm.
3. Use the Good Galerkin Approximation property.

The second step is impossible for the harmonic oscillator.

Concluding remarks

- Very few results about the structure of the attainable set.
- Some sufficient criterion for approximate controllability.
- Some reasonable estimates (L^1 norm, time, precision).
- Constructive methods (control and simulations are possible).

Future works

- Continuous spectrum.
- Time minimization: does there exist a minimal transfert time?
- Taking decoherence into account.