Bachelor Thesis

Large Time Solutions for Systems of One-Dimensional Hyperbolic Equations

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Zusammenfassung

Diese Thesis behandelt die Existenz von Lösungen des Anfangswertsproblems für Systeme von eindimensionalen hyperbolischen Gleichungen. Wir beginnen mit der Einführung von schwachen Lösungen für das Anfangswertproblem und leiten die sogenannte „jump condition“ her. Das ist eine wichtige Eigenschaft der schwachen Lösungen. Anschließend betrachten wir ein spezielles Anfangswertproblem, das Riemann-Problem. Wir zeigen, dass dieses eine schwache Lösung besitzt, wenn der Anfangswert gewisse Bedingungen erfüllt. Dafür gehen wir auf Eigenschaften von sogenannten „simple waves“ und „shock waves“ ein.
Wir nutzen unsere Ergebnisse anschließend, um das Glimm-Schema einzuführen. Dieses Schema ermöglicht uns unter gewissen Bedingungen schwache Lösungen für das Anfangswertproblem zu konstruieren.

Abstract

In this thesis the existence of solutions of the initial value problem for systems of one-dimensional hyperbolic equations is studied. We start by introducing weak solution for the initial value problem and deduce the jump condition. An important property, which weak solutions satisfy. Then a special initial value problem, the Riemann problem, is considered. We show, that the Riemann problem has a weak solution under some conditions. To prove that, some general theory of simple waves and shock waves will be presented.
We will use the existence of solutions of the Riemann problem to introduce Glimm's scheme. This scheme is used to construct weak solutions of an initial value problem under certain conditions.

Acknowledgments

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1 Introduction

The aim of this thesis is to present a way to constructed weak solutions of the initial value problem for systems of one dimensional hyperbolic equations using Glimm's scheme. By that the existence of weak solutions under some conditions for the initial value problem will be shown. This topic is usually fairly extensive in textbooks. So, another goal of this thesis is to focus on the essential parts. Therefore, important results needed to construct weak solutions will be introduced in the first two chapters of this thesis. In the third chapter Glimm's scheme will be presented based on chapter 19 of [Smo94]. Helpfull results from other references are added to provide more insight, but the aim is still to focus on the essential parts.

We will start with some basics. Therefore, in this chapter we will define the one dimensional system of hyperbolic equations and the corresponding initial value problem, which we want to study in this thesis. Also we will generalize the notion of solution for the initial value problem by introducing weak solutions. Finally we are going to deduce the so-called jump condition for the weak solutions.

1.1 One-Dimensional Systems of Conservation Laws

Let U be an open subset of $\mathbb{R}^N$, $N \in \mathbb{N}$, and $F : U \to \mathbb{R}^N$ a smooth function. F is called the flux function. We define the following system of equations

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} F(u(x,t)) = 0, \quad x \in \mathbb{R}, t > 0, \tag{1.1}$$

where

$$u(x,t) = \begin{pmatrix} u_1(x,t) \\ \vdots \\ u_N(x,t) \end{pmatrix}$$

is a vector-valued function from $\mathbb{R} \times \mathbb{R}_+$ into U. In this thesis $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.
In physics such systems are used to describe the conservation of matter. By reason of that (1.1) is called the general system of conservation laws in one space dimension. For example a one-dimensional system of conservation laws can be used to model compressible gas flow in one dimension [CM93]. We will give an example of a scalar conservations law used to model traffic flow.

Example 1.1. Let $N = 1$ and $\rho : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be the density of cars. For example ρ could be the number of cars per mile on a road. We can assume, that $0 \leq \rho(x,t) \leq \rho_{max}$, for some constant $\rho_{max} \in \mathbb{R}$. So $\rho(x,t) = 0$ if a section of the road is empty and $\rho(x,t) = \rho_{max}$ if a section of the road is completely full of cars. Also we can assume the velocity of the cars is bounded from above by some value v_{max}. For example this could be the speed limit. Then the scalar conservation law

$$\frac{\partial}{\partial t} \rho(x,t) + \frac{\partial}{\partial x} F(\rho(x,t)) = 0$$

with

$$F(\rho(x,t)) = v_{max} \rho(x,t) \left(1 - \frac{\rho(x,t)}{\rho_{max}}\right),$$

is a simple way to model the traffic flow on this road.

This example is taken from [Lev92], where one can find more details and other examples of one-dimensional conservation laws. We now return to the general system of one-dimensional conservation laws.

Definition 1.2. *Let $dF(u) \in \mathbb{R}^{N\times N}$ be the Jacobian matrix of $F(u)$. We call (1.1) hyperbolic, if for each $u \in U$ the matrix $dF(u)$ has N real and distinct eigenvalues $\lambda_1(u) < ... < \lambda_N(u)$.*

This definition is according to [Gli65]. From now on we assume that (1.1) is hyperbolic.
We will study the initial value problem, also called Cauchy problem, for (1.1). That means we want to find a function $u : \mathbb{R} \times \mathbb{R}_+ \to U$, which satisfies (1.1) and the initial condition

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $u_0 : \mathbb{R} \to U$ is a given function. The function $u_0(x)$ is called the initial data. From now on for simplicity we will call the initial value problem of (1.1) with initial data (1.2) the initial value problem (1.1), (1.2).
In the next example, taken from [Smo94], we will see, that in general the initial value problem (1.1), (1.2) does not have a continuously differentiable solution for all $t \in \mathbb{R}_+$. For the example we need the following definition from [GR21].

Definition 1.3. *Let C_k be the integral curves of the differential systems*

$$\frac{dx}{dt} = \lambda_k(u(x, t))$$

where $k \in \{1, ..., N\}$. The curves C_k are called the characteristic curves of the k-th field.

Example 1.4. Consider the following scalar conservation law

$$\frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x}(\frac{1}{2}(u(x, t))^2) = 0, \quad x \in \mathbb{R}, t > 0. \tag{1.3}$$

This equation is called Burger's equation. We can write (1.3) to get

$$\frac{\partial}{\partial t} u(x, t) + u(x, t)\frac{\partial}{\partial x} u(x, t) = 0.$$

We will consider the initial value problem for 1.3 with initial data

$$u_0(x) = \begin{cases} 1 & \text{if } x < 0 \\ 1 - x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}. \tag{1.4}$$

Suppose a function $u : \mathbb{R} \times \mathbb{R}_+ \to U$ is a continuously differentiable solution of the initial value problem (1.3), (1.4), defined for all $t > 0$. For any $x_0 \in \mathbb{R}$ and $t_0 \geq 0$ consider the characteristic curve through (x_0, t_0). Since $N = 1$ the characteristic curve through (x_0, t_0) is defined by $(x(t), t)$, where $x : \mathbb{R}_+ \to \mathbb{R}$ is the solution of the initial value problem for

$$\frac{dx}{dt} = u(x, t),$$

with the initial condition

$$x(t_0) = x_0.$$

For all $x_0 \in \mathbb{R}$ and $t_0 \geq 0$ the curve exists at least in some small time interval. We have

$$\frac{d}{dt} u(x(t), t) = \frac{\partial}{\partial x} u(x(t), t)\frac{d}{dt} x(t) + \frac{\partial}{\partial t} u(x(t), t) = u(x(t), t)\frac{\partial}{\partial x} u(x(t), t) + \frac{\partial}{\partial t} u(x(t), t) = 0$$

So the function u is constant along the characteristic curves and thus the characteristic curve are straight lines in the x-t-plane. The constant slopes of the curves depend on the initial data. Figure 1 depicts some of these curves for $0 \leq t \leq 1$. We can see that characteristic curves intersect at the point $(1,1)$. Since for example the characteristic curves through the points $(0,0)$ and $(1,0)$ meet the point $(1,1)$, the solution u should have the value $u(1,1) = 0$ and $u(1,1) = 1$. Therefore the solution u cannot be continuous at that point. This hold for all points, at which at least two characteristic curves intersect. So we cannot define a continuous solution of the initial value problem for $t \geq 1$.

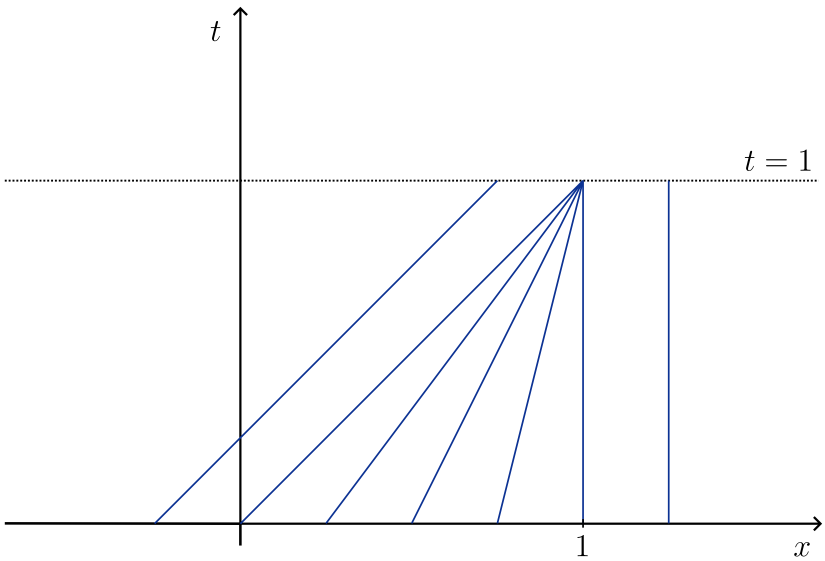

Figure 1: The characteristic curves of the initial value problem (1.3), (1.4)

In the example we could see, that in general we cannot assume that the initial value problem (1.1), (1.2) has a continuous solution for all $t \geq 0$. If we want to define solutions for all $t \geq 0$ we need to generalize our notion of solution in some way. We will do that similarly to [GR21].

Definition 1.5. *Let $C_0^1(\mathbb{R} \times \mathbb{R}_+)^N$ be the space of all continuously differentiable function with compact support in $\mathbb{R} \times \mathbb{R}_+$.*

Let $u_0 \in L^\infty_{loc}(\mathbb{R})^N$, where $L^\infty_{loc}(\mathbb{R})^N$ is the space of locally bounded measurable functions. Now assume u is a continuously differentiable solution of the initial value problem (1.1), (1.2). Let $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}_+)^N$. We multiply (1.1) with ϕ and integrate over $\mathbb{R} \times \mathbb{R}_+$. Then we get

$$\int_{\mathbb{R}} \int_0^\infty (\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} F(u(x,t))) \cdot \phi(x,t) dt dx = 0.$$

Note that the dot $\cdot$ denotes the Euclidean inner product on $\mathbb{R}^N$. We can use Fubini's theorem [Bau92] and integration by parts to obtain

$$\int_0^\infty \int_{-\infty}^\infty u(x,t) \cdot \frac{\partial}{\partial t}\phi(x,t) + F(u(x,t)) \cdot \frac{\partial}{\partial x}\phi(x,t)) dx dt + \int_{-\infty}^\infty u_0(x) \cdot \phi(x,0) dx = 0. \quad (1.5)$$

Definition 1.6. *Let $u_0 \in L^\infty_{loc}(\mathbb{R})^N$. We call a function $u \in L^\infty_{loc}(\mathbb{R} \times \mathbb{R}_+)^N$ a weak solution of the initial value problem (1.1),(1.2) if $u(x,t) \in U$ almost everywhere and u satisfies (1.5) for all $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}_+)^N$.*

A continuously differentiable solution of the initial value problem (1.1), (1.2) is a weak solution. That follows directly from the construction of (1.5). Now assume $u : \mathbb{R} \times \mathbb{R}_+ \to U$ is a continuously differentiable weak solution of the initial value problem (1.1), (1.2). Let $\phi \in C_0^1(\mathbb{R} \times (0,\infty))^N$, then with (1.5) we get

$$\int_0^\infty \int_{-\infty}^\infty (\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} F(u(x,t))) \cdot \phi(x,t) dx dt = 0.$$

Since this holds for all $\phi \in C_0^1(\mathbb{R} \times (0,\infty))^N$, we have

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} F(u(x,t)) = 0$$

for all $(x,t) \in \mathbb{R} \times (0,\infty)$. Now we can multiply this by a function $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}_+)^N$ and again use integration by parts and Fubini's theorem to get

$$\int_0^\infty \int_{-\infty}^\infty u(x,t) \cdot \frac{\partial}{\partial t} \phi(x,t) + F(u(x,t)) \cdot \frac{\partial}{\partial x} \phi(x,t) dx dt + \int_{-\infty}^\infty u(x,0) \cdot \phi(x,0) dx = 0.$$

Comparing this with (1.5) gives us

$$\int_{-\infty}^\infty (u(x,0) - u_0(x)) \cdot \phi(x,0) dx = 0.$$

Again this holds for all $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}_+)^N$. Thus u is a continuously differentiable solution of the initial value problem (1.1), (1.2). We can conclude: A continuously differentiable function $u : \mathbb{R} \times \mathbb{R}_+ \to U$ is a weak solution of the initial value problem (1.1), (1.2) if and only if u is a solution of the initial value problem in the classical sense.

1.2 The Jump Condition

We shall note that a weak solution of the initial value problem (1.1), (1.2) can be discontinuous. This is important, since, as we seen before, solutions of the initial value problem are in general not continuous for all $t \in \mathbb{R}_+$. Let Ω be a smooth curve and assume Ω has a parametrization of the form $(x(t), t)$, where $x : I \to \mathbb{R}$ is a smooth function and I an open interval in $\mathbb{R}_+$. Assume a function $u : \mathbb{R} \times \mathbb{R}_+ \to U$ is a weak solution of the initial value problem (1.1), (1.2). Suppose u has a discontinuity across the curve Ω, u has well-defined limits on both side of Ω and is continuously differentiable on either side of the curve Ω. Such a discontinuity is called a jump discontinuity, according to [Smo94]. We now are going to investigate properties of such a discontinuity. The next part is based on [GR21] and [Eva98].
Let $P \in \mathbb{R} \times \mathbb{R}_+$ be a point on the curve Ω and $V \subseteq \mathbb{R} \times (0,\infty)$ an open ball centered at P. Let V be small enough, such that Ω separates V into two components. Let V_r and V_l be the components of V separated by Ω as depicted in Figure 2. We assume V_r lies on the right side of the curve Ω and V_l on the left. V_r and V_l are both open subsets of U.
Let $\phi \in C_0^1(V)^N$, then with (1.5) we get

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty u(x,t) \cdot \frac{\partial}{\partial t} \phi(x,t) + F(u(x,t)) \cdot \frac{\partial}{\partial x} \phi(x,t) dx dt + \int_{-\infty}^\infty u_0(x) \cdot \phi(x,0) dx \\ &= \iint_V u(x,t) \cdot \frac{\partial}{\partial t} \phi(x,t) + F(u(x,t)) \cdot \frac{\partial}{\partial x} \phi(x,t) dx dt, \end{aligned}$$

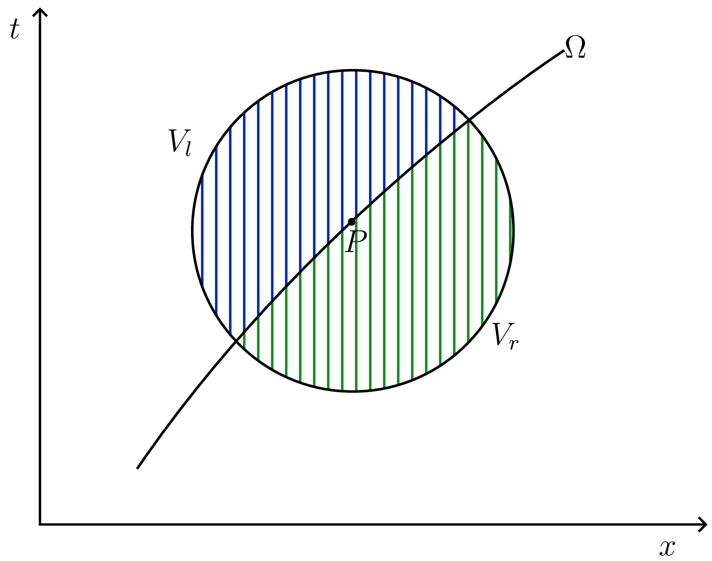

Figure 2: The open ball V centred at P.

since $\phi(x,t) = 0$ for all $(x,t) \notin V$. This leads to

$$\begin{aligned}
0 &= \iint_V u(x,t) \cdot \frac{\partial}{\partial t}\phi(x,t) + F(u(x,t)) \cdot \frac{\partial}{\partial x}\phi(x,t)dxdt \\
&= \iint_{V_r} u(x,t) \cdot \frac{\partial}{\partial t}\phi(x,t) + F(u(x,t)) \cdot \frac{\partial}{\partial x}\phi(x,t)dxdt \\
&+ \iint_{V_l} u(x,t) \cdot \frac{\partial}{\partial t}\phi(x,t) + F(u(x,t)) \cdot \frac{\partial}{\partial x}\phi(x,t)dxdt.
\end{aligned}$$

Now we define

$$\begin{aligned}
u_r(x(t),t) &= \lim_{\epsilon \to 0^+} u(x(t)+\epsilon, t) \\
u_l(x(t),t) &= \lim_{\epsilon \to 0^-} u(x(t)+\epsilon, t).
\end{aligned}$$

Seeing that $\phi(x,t) = 0$ if $(x,t) \in \partial V$ we can use Green's formula [Olv13] to get

$$\begin{aligned}
&\iint_{V_r} u(x,t) \cdot \frac{\partial}{\partial t}\phi(x,t) + F(u(x,t)) \cdot \frac{\partial}{\partial x}\phi(x,t)dxdt \\
&= -\iint_{V_r} (\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}F(u(x,t))) \cdot \phi(x,t)dxdt \\
&\quad - \int_{\Omega \cap V} (u_r(x,t)\nu_2(x,t) + F(u_r(x,t))\nu_1(x,t)) \cdot \phi(x,t)d(\Omega \cap V).
\end{aligned}$$

Here $\nu(x,t) = (\nu_1(x,t), \nu_2(x,t))^T$ is the unit normal to the curve Ω pointing from V_l into V_r. Considering that u is continuously differentiable in V_r and V_l we get

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}F(u(x,t)) = 0,$$

for all $(x,t) \in V_r$ or $(x,t) \in V_l$. So we obtain

$$\begin{aligned}
&\iint_{V_r} u(x,t) \cdot \frac{\partial}{\partial t}\phi(x,t) + F(u(x,t))) \cdot \frac{\partial}{\partial x}\phi(x,t)dxdt \\
&= -\int_{\Omega \cap V} (u_r(x,t)\nu_2(x,t) + F(u_r(x,t))\nu_1(x,t)) \cdot \phi(x,t)d(\Omega \cap V)
\end{aligned}$$

and in the same way

$$\iint_{V_l} u(x,t) \cdot \frac{\partial}{\partial t}\phi(x,t) + F(u(x,t))) \cdot \frac{\partial}{\partial x}\phi(x,t)dxdt$$
$$= \int_{\Omega \cap V} (u_l(x,t)\nu_2(x,t) + F(u_l(x,t))\nu_1(x,t)) \cdot \phi(x,t)d(\Omega \cap V).$$

This gives us

$$0 = \int_{\Omega \cap V} [(u_l(x,t) - u_r(x,t))\nu_2(x,t) + (F(u_l(x,t)) - F(u_r(x,t)))\nu_1(x,t)] \cdot \phi(x,t)d(\Omega \cap V).$$

Recall that this holds for all $\phi \in C_0^1(V)^N$, so we get

$$0 = (u_l(x,t) - u_r(x,t))\nu_2(x,t) + (F(u_l(x,t)) - F(u_r(x,t)))\nu_1(x,t)$$

for any point (x,t) on $\Omega \cap V$. Since Ω is defined by $(x(t),t)$ the unit normal to the curve Ω pointing in the direction of V_r is given by

$$\nu(x(t),t) = \begin{pmatrix} \nu_1(x(t),t) \\ \nu_2(x(t),t) \end{pmatrix} = \frac{1}{\sqrt{1 + \frac{d}{dt}x(t)^2}} \begin{pmatrix} 1 \\ -\frac{d}{dt}x(t) \end{pmatrix}.$$

So we get

$$F(u_l(x,t)) - F(u_r(x,t)) = (u_l(x,t) - u_r(x,t))\frac{d}{dt}x(t) \tag{1.6}$$

for all $(x,t) \in \Omega \cap V$. Now for a point $P' = (x(t'),t') \in \Omega \cap V$, $t' \in I$, define

$$[u] = u_l(P') - u_r(P')$$
$$[F(u)] = F(u_l(P')) - F(u_r(P'))$$
$$s = \frac{d}{dt}x(t').$$

The value s is called the speed of the discontinuity at the point P'. With that we can write (1.6) at a point P' as

$$[F(u)] = s[u] \tag{1.7}$$

The equation (1.7) is called the jump condition. The following theorem states an important relation between weak solutions and the jump condition. The theorem is a simplified version of Theorem 4.1 in [GR21].

Theorem 1.7. *Let $u : \mathbb{R} \times \mathbb{R}_+ \to U$ be a piecewise continuous differentiable function. Suppose u only has a finite number of jump discontinuities across smooth curves. The curves can be represented parametrically by $(x(t),t)$, where x is a smooth function from some open Interval $I \subseteq (0,\infty)$ into $\mathbb{R}$. Then u satisfies*

$$\int_0^\infty \int_{-\infty}^\infty u(x,t) \cdot \frac{\partial}{\partial t}\phi(x,t) + F(u(x,t)) \cdot \frac{\partial}{\partial x}\phi(x,t)dxdt = 0$$

for all $\phi \in C_0^1(\mathbb{R} \times (0,\infty))^N$ if and only if

(i) u is a solution of (1.1) in the domains, where u is continuously differentiable;

(ii) u satisfies the jump condition (1.7) at every point on every curve.

A domain is an open and connected set. At the end of this chapter we want to give an example, taken from [GR21], that shows another important property of weak solutions: In general we cannot expect that weak solutions of the initial value problem (1.1), (1.2) are unique.

Example 1.8. Again consider an initial value problem for Burger's equation

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}(\frac{1}{2}(u(x,t))^2) = 0$$

with initial data

$$u_0(x) = \begin{cases} u_1 & \text{if } x < 0 \\ u_2 & \text{if } x > 0 \end{cases}$$

where $u_1, u_2 \in \mathbb{R}$. Then

$$u(x,t) = \begin{cases} u_1 & \text{if } x < st \\ u_2 & \text{if } x > st \end{cases}$$

is a weak solution of this initial value problem if $s = \frac{1}{2}(u_1 + u_2)$. This follows from theorem 1.7, since

$$\frac{1}{2}(u_1 + u_2)(u_1 - u_2) = \frac{1}{2}u_1^2 - \frac{1}{2}u_2^2.$$

So the solution satisfies the jump condition (1.7) along the discontinuity. Now choose an $a \in \mathbb{R}$ with $a \geq u_1$ and $a \geq -u_2$, then

$$u(x,t) = \begin{cases} u_1 & \text{if } x < s_1 t \\ -a & \text{if } s_1 t < x < 0 \\ a & \text{if } 0 < x < s_2 t \\ u_2 & \text{if } s_2 t < x \end{cases}$$

with $s_1 = \frac{1}{2}(u_1 - a)$ and $s_1 = \frac{1}{2}(u_2 + a)$ is a weak solution of the initial value problem. Again this follows from theorem 1.7. So we can see, the weak solution of the initial value problem is not only not unique, but there are infinitely many weak solutions since there are infinitely many options to choose a.

2 The Riemann Problem

In this chapter we are going to introduce a special initial value problem for (1.1), the so-called Riemann problem. We will show, that under certain conditions a weak solution of the Riemann problem exists. Therefore we need to define shock waves and simple waves and deduce some properties of them.

2.1 Definition

First we need to define the Riemann problem for (1.1). Let $u_l, u_r \in U$. We will call the elements of U henceforth states. The initial value problem for (1.1) with initial data

$$u_0(x) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases} \tag{2.1}$$

is called the Riemann problem for (1.1). Remember we assumed that (1.1) is hybprbolic, so the Jacobian matrix $dF(u)$ of $F(u)$ has N real and distinct eigenvalues $\lambda_1(u) < ... < \lambda_N(u)$ for all $u \in U$. Corresponding to each eigenvalue we have a right eigenvector $r_k(u) \in \mathbb{R}^N$ defined by

$$dF(u)r_k(u) = \lambda_k(u)r_k(u)$$

and a left eigenvector $l_k(u) \in \mathbb{R}^N$ defined by

$$l_k(u)^T dF(u) = \lambda_k(u)l_k(u)^T$$

for all $k \in \{1, ..., N\}$.

Theorem 2.1. *Let $dF(u)$ be the Jacobian matrix of $F(u)$.*

(i) The eigenvalues $\lambda_1(u) < ... < \lambda_N(u)$ of $dF(u)$ depend smoothly on $u \in U$.

(ii) The left eigenvector $l_k(u)$ and the right eigenvector $r_k(u)$ depend smoothly on $u \in U$ for all $k \in \{1, ..., N\}$.

Remark 2.2. The left and right eigenvectors of the matrix $dF(u)$ satisfy following properties:

(i) $l_k(u)^T \cdot r_j(u) = 0$ for all $k, j \in \{1, ..., N\}$ with $k \neq j$.

(ii) $l_k(u)^T \cdot r_k(u) \neq 0$ for all $k \in \{1, ..., N\}$.

The theorem and the content of the remark are from [Eva98]. We will continue with some definitions.

Definition 2.3. *Let $k \in \{1, ..., N\}$. The pair $(\lambda_k(u), r_k(u))$ is called the k-th characteristic field.*

Definition 2.4. *The k-th characteristic field $(\lambda_k(u), r_k(u))$ is called genuinely nonlinear if*

$$\nabla\lambda_k(u) \cdot r_k(u) \neq 0$$

for all $u \in U$. We call the k-th characteristic field linearly degenerate if

$$\nabla\lambda_k(u) \cdot r_k(u) = 0$$

for all $u \in U$.

The definitions and the following remark are from [GR21].

Remark 2.5. For the scalar case $N = 1$ we have $\lambda_1(U) = \frac{d}{du}F(u)$ and $r_1(u) = 1$. So

$$\nabla\lambda_1(u) \cdot r_k(u) = \frac{d^2}{du^2}F(u) \cdot 1 = \frac{d^2}{du^2}F(u)$$

Thus in the scalar case the characteristic field is genuinely nonlinear if $\frac{d^2}{du^2}F(u) \neq 0$ for all $u \in U$.

Regarding the following, we assume that (1.1) is genuinely nonlinear in each characteristic field. That is because we only need that case in the next chapter. However it can be proven, that weak solutions of the Riemann problem for (1.1) exist under some conditions if each characteristic field is either genuinely nonlinear or linearly degenerate. For this case we refer to [HR15] or [GR21]. For simplicity we normalize $r_k(u)$ and $l_k(u)$, such that

$$\nabla\lambda_k(u) \cdot r_k(u) = 1$$
$$l_k(u)^T \cdot r_k(u) = 1$$

for all $k \in \{1, ..., N\}$. This normalization will be useful in the next parts.

2.2 Simple Waves

In this section we will develop some general theory of simple-waves. We start with some definitions.

Definition 2.6. *Let $k \in \{1, ..., N\}$. We call a function $w : U \to \mathbb{R}$ a k-Riemann invariant if*

$$\nabla w(u) \cdot r_k(u) = 0$$

for all $u \in U$.

Definition 2.7. *A continuously differentiable solution of the initial value problem* (1.1), (1.2) *in a domain D is called a k-simple wave, $k \in \{1, ..., N\}$, if all k-Riemann invariants are constant in D.*

These definitions are from [Lax57]. Before we continue we want to state some properties of k-simple waves and k-Riemann invariants.

Theorem 2.8. *For every $k \in \{1, ..., N\}$ exist $(N - 1)$ k-Riemann invariants, whose gradients are linearly independent.*

Theorem 2.9. *Assume u is a k-simple wave, $k \in \{1, ..., N\}$, then the characteristic curves of the k-th field are straight lines and u is constant along the characteristic curves.*

These theorems are from [Smo94].
Now we will define k-centered simple waves, $k \in \{1, ..., N\}$, similar to [GR21]. Let $u_l, u_r \in U$ and let u be the solution of the Riemann problem for (1.1) with initial data

$$u_0(x) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0. \end{cases}$$

Suppose u has the form $u(x,t) = v(\frac{x}{t})$ for $t > 0$, where $v : \mathbb{R} \to U$. Such a solution is called a self similar weak solution. Let D be a domain in the x-t-plane, where u and so v is continuously differentiable. Then we have

$$\frac{\partial}{\partial t}u(x,t) + dF(u(x,t))\frac{\partial}{\partial x}u(x,t) = 0,$$

since u satisfies (1.1) in D. With $u(x,t) = v(\frac{x}{t})$ we get

$$-\frac{x}{t^2}v'(\frac{x}{t}) + \frac{1}{t}dF(v(\frac{x}{t}))v'(\frac{x}{t}) = 0$$

and so

$$(dF(v(\frac{x}{t})) - (\frac{x}{t})I)v'(\frac{x}{t}) = 0.$$

By setting $\gamma = \frac{x}{t}$ we obtain

$$(dF(v(\gamma)) - \gamma I)v'(\gamma) = 0.$$

Therefore one of the following two condition must hold for v:

(i) $v'(\gamma) = 0$

(ii) $v'(\gamma) = \alpha(\gamma)r_k(v(\gamma))$ and $\lambda_k(v(\gamma)) = \gamma$ for some function $\alpha : \mathbb{R} \to \mathbb{R}$ and a $k \in \{1, ..., N\}$

With the normalization from before we get

$$1 = \frac{d}{d\gamma}\lambda_k(v(\gamma)) = \nabla\lambda_k(v(\gamma)) \cdot v'(\gamma) = \alpha(\gamma)\nabla\lambda_k(v(\gamma)) \cdot r_k(v(\gamma)) = \alpha(\gamma).$$

So the conditions simplify to

(i) $v'(\gamma) = 0$

(ii) $v'(\gamma) = r_k(v(\gamma))$ and $\lambda_k(v(\gamma)) = \gamma$ for a $k \in \{1, ..., N\}$

Now let $v : \mathbb{R} \to U$ and $v_c : \mathbb{R} \to U$ be two functions. We assume v_c is continuously differentiable satisfying the second condition for a fixed $k \in \{1, ..., N\}$. Also we suppose $v_c(\lambda_k(u_l)) = u_l$ and $v_c(\lambda_k(u_r)) = u_r$ with $\lambda_k(u_l) \le \lambda_k(u_r)$ for some states $u_l, u_r \in U$. Then we can define a function $u : \mathbb{R} \times \mathbb{R}_+ \to U$ by

$$u(x,t) = v(\frac{x}{t}) = \begin{cases} u_l & \text{if } \frac{x}{t} < \lambda_k(u_l) \\ v_c(\frac{x}{t}) & \text{if } \lambda_k(u_l) \le \frac{x}{t} \le \lambda_k(u_r) \\ u_r & \text{if } \frac{x}{t} > \lambda_k(u_r) \end{cases}$$

for $t > 0$ and

$$u(x,0) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases} \tag{2.2}$$

for $t = 0$. Thanks to our findings before, u is a self similar weak solution of the Riemann problem for (1.1) with the initial condition (2.2). The function u in the x-t-plane is depicted in Figure 3.

Lemma 2.10. *The k-Riemann invariants are constant for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ with $\lambda_k(u_l) \le \frac{x}{t} \le \lambda_k(u_r)$, $t > 0$.*

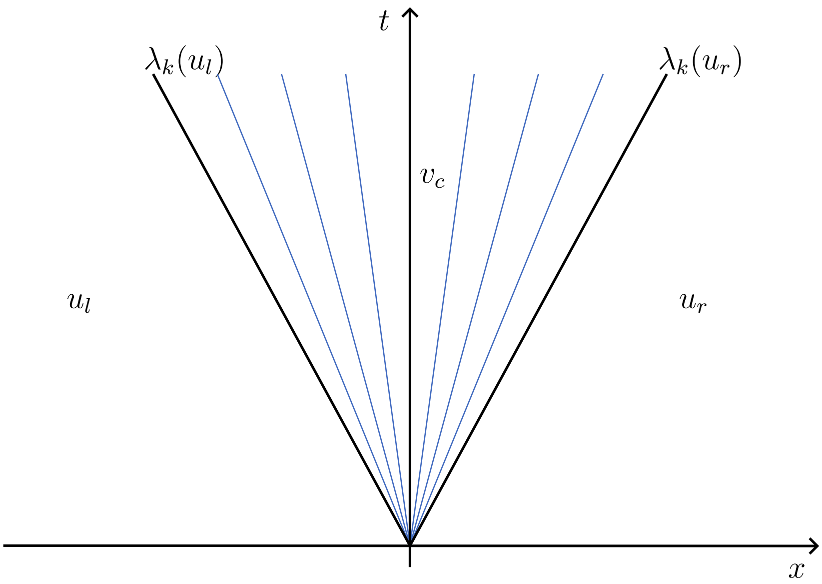

Figure 3: The function u in the x-t-plane

Proof. Let w be a k-Riemann invariant then

$$\frac{d}{d\gamma}w(v_c(\gamma)) = \nabla w(v_c(\gamma)) \cdot \frac{d}{d\gamma}v_c(\gamma) = \nabla w(v_c(\gamma)) \cdot r_k(v_c(\gamma)) = 0$$

So since $u(x,t) = v_c(\frac{x}{t})$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ with $\lambda_k(u_l) \leq \frac{x}{t} \leq \lambda_k(u_r)$, $t > 0$, the k-Riemann invariants are constant for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ with $\lambda_k(u_l) \leq \frac{x}{t} \leq \lambda_k(u_r)$, $t > 0$. □

Therefore if $\lambda_k(u_l) < \lambda_k(u_r)$ u is a k-simple wave in the domain defined by $\lambda_k(u_l) < \frac{x}{t} < \lambda_k(u_r)$, $t > 0$, because u is a continuously differentiable solution in that domain. If $\lambda_k(u_l) = \lambda_k(u_r)$ then $u_r = u_l$ and so $u(x,t) = u_l$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. Hence u is a k-simple wave in $\mathbb{R} \times (0, \infty)$. This justifies the next definition.

Definition 2.11. *We call such a self similar weak solution of the Riemann problem of* (1.1) *with the initial condition* (2.2) *a k-centered simple wave connecting the state u_l to u_r on the right.*

Suppose we have a state $u_l \in U$ given, then we want to find all states $u_r \in U$ in a neighborhood of u_l, that can be connected to u_l on the right by a k-centered simple wave. Therefore we will prove the following theorem. The proof is based on [GR21].

Theorem 2.12. *Let $u_l \in U$ be a given state, then there exists a $\gamma_0 > 0$ and a smooth function $R^k_{u_l} : [-\gamma_0, \gamma_0] \to U$, such that the set $\{R^k_{u_l}(\varepsilon) : \varepsilon \in [0, \gamma_0]\}$ consists of all states in a neighborhood of u_l which can be connected to u_l on the right by a k-centered simple wave.*

Proof. Let $u_l \in U$, $k \in \{1, ..., N\}$ and consider the initial value problem for

$$\frac{d}{d\gamma}v_k(\gamma) = r_k(v_k(\gamma))$$

with the initial condition

$$v_k(\lambda_k(u_l)) = u_l.$$

According to [MM54], there exists a unique solution $v_k(\gamma)$ on an interval $[\lambda_k(u_l)-\gamma_0, \lambda_k(u_l)+\gamma_0]$ for a $\gamma_0 > 0$ sufficiently small. Since

$$\nabla\lambda_k(v_k(\gamma)) \cdot r_k(v_k(\gamma)) = 1,$$

we obtain

$$\frac{d}{d\gamma}\lambda_k(v_k(\gamma)) = \nabla\lambda_k(v_k(\gamma)) \cdot \frac{d}{d\gamma}v_k(\gamma) = \nabla\lambda_k(v_k(\gamma)) \cdot r_k(v_k(\gamma)) = 1.$$

Hence

$$\lambda_k(v_k(\gamma)) - \lambda_k(v_k(\lambda_k(u_l)) = \gamma - \lambda_k(u_l)$$

and so

$$\lambda_k(v_k(\gamma)) = \gamma.$$

Let $u_r \in \{v_k(\gamma) : \gamma \geq 0\}$, then $\lambda_k(u_r) \geq \lambda_k(u_l)$ and $v_k(\lambda_k(u_r)) = u_r$. By our findings before the function $u : \mathbb{R} \times \mathbb{R}_+ \to U$ defined by

$$u(x,t) = \begin{cases} u_l & \text{if } \frac{x}{t} < \lambda_k(u_l) \\ v_k(\frac{x}{t}) & \text{if } \lambda_k(u_l) \leq \frac{x}{t} \leq \lambda_k(u_r) \\ u_r & \text{if } \frac{x}{t} > \lambda_k(u_r) \end{cases}$$

for $t > 0$ and

$$u(x,0) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}$$

for $t = 0$ is a weak solution of the Riemann problem for (1.1) with initial data

$$u_0(x) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0. \end{cases}$$

So the state u_r can be connected to u_l on the right by a k-centered simple wave. Therefore the set $\{v_k(\gamma) : \gamma \geq \lambda_k(u_l)\}$ consists of states, which can be connected to u_l on the right by a k-centered simple wave. If $\gamma < 0$ we have $\lambda_k(v_k(\gamma)) < \lambda_k(u_l)$, so $v_k(\gamma)$ cannot be connected to u_l on the right by a k-centered simple wave. In the set

$$\{u \in U : \lambda_k(v_k(\lambda_k(u_l) - \gamma_0)) \leq \lambda_k(u) \leq \lambda_k(v_k(\lambda_k(u_l) + \gamma_0))\}$$

are no other states, that can be connected to u_l on the right by a k-centered simple wave. Assume there is a state

$$u' \in \{u \in U : \lambda_k(v_k(\lambda_k(u_l) - \gamma_0)) \leq \lambda_k(u) \leq \lambda_k(v_k(\lambda_k(u_l) + \gamma_0))\}$$

and $u' \notin \{v_k(\gamma) : \gamma \geq \lambda_k(u_l)\}$, that can be connected to u_l on the right by a k-centered simple wave. Then per definition $\lambda_k(u') \geq \lambda_k(u_l)$ and there exists a function $\tilde{v}_k(\gamma)$ with $\tilde{v}_k(\lambda_k(u_l)) = u_l$ and $\tilde{v}_k(\gamma') = u'$ for some $\gamma' \geq \lambda_k(u_l)$. Also

$$\frac{d}{d\gamma}\tilde{v}_k(\gamma) = r_k(\tilde{v}_k(\gamma)).$$

Thus, according to [MM54], $\tilde{v}_k = v_k$. This is a contradiction to our assumption. So the set $\{v_k(\gamma) : \gamma \geq \lambda_k(u_l)\}$ consists of all states in a neighborhood of u_l, which can be connected to u_l on the right by a k-centered simple wave. Now define $R^k_{u_l} : [0, \gamma_0] \to U$ by

$$R^k_{u_l}(\varepsilon) = v_k(\varepsilon + \lambda_k(u_l)).$$

This completes the proof. □

We conclude this subsection by stating an important result of this theorem.

Corollary 2.13. *The derivatives of* $R^k_{u_l}(\varepsilon)$ *are given by*

(i) $\frac{d}{d\varepsilon}R^k_{u_l}(\varepsilon) = r_k(R^k_{u_l}(\varepsilon))$

(ii) $\frac{d^2}{d\varepsilon^2}R^k_{u_l}(\varepsilon) = \frac{d}{d\varepsilon}r_k(R^k_{u_l}(\varepsilon))$

Proof. This follows directly from the definition of $R^k_{u_l}(\varepsilon)$. □

Remark 2.14. For $\varepsilon = 0$ we get

(i) $\frac{d}{d\varepsilon}R^k_{u_l}(0) = r_k(u_l)$

(ii) $\frac{d^2}{d\varepsilon^2}R^k_{u_l}(0) = \nabla r_k(u_l) \cdot r_k(u_l)$

2.3 Shock Waves

In this chapter we want to deduce a similar result for k-shock waves, $k \in \{1, ..., N\}$. Let $u_r, u_l \in U$ be two given states and $s \in \mathbb{R}$, then we define a function $u : \mathbb{R} \times \mathbb{R}_+ \to U$ by

$$u(x,t) = \begin{cases} u_l & \text{if } x < st \\ u_r & \text{if } x > st. \end{cases} \tag{2.3}$$

The function u is a weak solution of the Riemann problem for (1.1) with initial data

$$u_0(x) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0. \end{cases}$$

if s satisfies the jump condition

$$s(u_r - u_l) = F(u_l) - F(u_r).$$

According to [GR21], we call such a weak solution a discontinuity wave. The function u in the x-t-plane is depicted in Figure 4. If for some states $u_l, u_r \in U$ the Riemann problem for (1.1) with initial date

$$u_0(x) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0. \end{cases}$$

has such a solution, we say u_l can be connected to u_r on the right by a discontinuity wave. For a fixed state u_l we now want to find all states, that can be connected to u_l on the right by a discontinuity wave. Therefore we are going to prove the following theorem. The proof is based on [GR21] and [Eva98].

Theorem 2.15. *Let* $u_l \in U$, *then there exist* N *smooth functions* $S^k_{u_l} : E \to U$, $k \in \{1, ..., N\}$, *where* E *is a neighborhood of* 0 *small enough. The set* $\{S^k_{u_l}(\xi) : \xi \in E, k \in \{1, ..., N\}\}$ *consists of all states* u_r *near* u_l, *such that there exists a real number* s *with*

$$s(u_r - u_l) = F(u_l) - F(u_r).$$

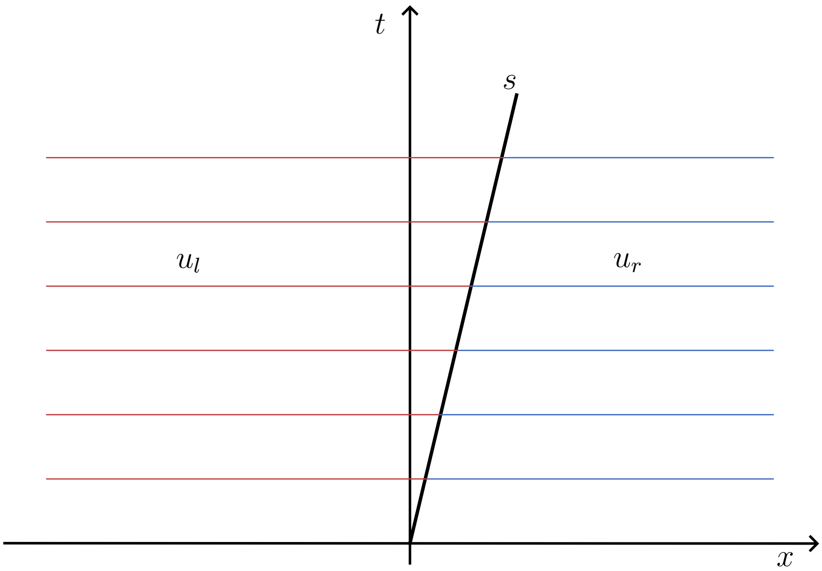

Figure 4: The discontinuity wave connecting u_r to u_l on the right in the x-t-plane.

Proof. Let $u_l \in U$ be a fixed state. For any state $u \in U$ we can write

$$F(u) - F(u_l) = \int_0^1 \frac{d}{d\sigma} F(u_l + \sigma(u - u_l)) d\sigma$$
$$= \int_0^1 dF(u_l + \sigma(u - u_l)) d\sigma (u - u_l).$$

Set

$$A_{u_l}(u) = \int_0^1 dF(u_l + \sigma(u - u_l)) d\sigma,$$

then the jump condition can be written as

$$A_{u_l}(u)(u - u_l) = s(u - u_l). \tag{2.4}$$

Note that

$$A_{u_l}(u_l) = \int_0^1 dF(u_l + \sigma(u_l - u_l)) d\sigma = dF(u_l)$$

and so $A_{u_l}(u_l)$ has N real and distinct eigenvalues $\lambda_1(u_l) < ... < \lambda_N(u_l)$. Since $dF(u)$ is smooth, there is a neighborhood $\mathcal{N}_{u_l} \subseteq U$ of u_l, such that smooth functions $\lambda_{k,u_l} : \mathcal{N}_{u_l} \to \mathbb{R}$, $r_{k,u_l} : \mathcal{N}_{u_l} \to \mathbb{R}^N$, $l_{k,u_l} : \mathcal{N}_{u_l} \to \mathbb{R}^N$ exist, with

$$\lambda_{k,u_l}(u_l) = \lambda_k(u_l), \qquad r_{k,u_l}(u_l) = r_k(u_l), \qquad l_{k,u_l} = l_k(u_l)$$

and

$$A_{u_l}(u) r_{k,u_l}(u) = \lambda_{k,u_l}(u) r_{k,u_l}(u)$$
$$l_{k,u_l}(u)^T A_{u_l}(u) = \lambda_{k,u_l}(u) l_{k,u_l}(u)^T$$

for $k \in \{1, ..., N\}$. A state $u \in \mathcal{N}_{u_l}$ satisfies (2.4) if and only if a $k \in \{1, ..., N\}$ exists, such that $s = \lambda_{k,u_l}(u)$ and $\alpha(u - u_l) = r_{k,u_l}(u)$ for an $\alpha \in \mathbb{R}$. Since

$$l_{k,u_l}(u)^T r_{j,u_l} = 0$$

for $j \in \{1, ..., N\}$ with $k \neq j$, we have $\alpha(u - u_l) = r_{k,u_l}(u)$ if and only if

$$l_{j,u_l}(u)^T(u - u_l) = 0$$

for all $j \in \{1, ..., N\}$ with $k \neq j$. This gives us $N - 1$ equations for each fixed k, which u needs to satisfy. We define

$$G_k(u) = M_k(u)(u - u_l)$$

with

$$M_k(u) = \begin{pmatrix} l_{1,u_l}(u)^T \\ \vdots \\ l_{k-1,u_l}(u)^T \\ l_{k+1,u_l}(u)^T \\ \vdots \\ l_{N,u_l}(u)^T \end{pmatrix}.$$

So $G_k(u)$ is a smooth function from $\mathcal{N}_{u_l}$ into $\mathbb{R}^N$. Let $dG_k(u)$ be the Jacobian matrix of $G_k(u)$, then

$$dG_k(u_l) = \begin{pmatrix} l_{1,u_l}(u_l)^T \\ \vdots \\ l_{k-1,u_l}(u_l)^T \\ l_{k+1,u_l}(u_l)^T \\ \vdots \\ l_{N,u_l}(u_l)^T \end{pmatrix}$$

Since $l_{j,u_l}(u_l) = l_j(u_l)$ for $j \in \{1, ..., N\}$, $dG_k(u_l)$ has rank $(N - 1)$. Thus there exists a $p \in \{1, ..., N\}$, such that $dG_k(u_l)$ without the pth column is a $(N - 1) \times (N - 1)$ matrix with full rank. Also we have $G_k(u_l) = 0$. Let $u_l = (u_l^1, ..., u_l^N)^T$ and $u = (u^1, ..., u^N)^T$, then by the implicit function theorem [Dei21] there are open neighborhoods $\mathcal{M}_1$ of u_l^p and $\mathcal{M}_2$ of $(u_l^1, ..., u_l^{p-1}, u_l^{p+1}, ..., u_l^N)^T$ and a unique smooth function

$$H_{u_l}^k : \mathcal{M}_1 \to \mathcal{M}_2$$

with

$$H_{u_l}^k(\gamma) = (H_{u_l}^{k,1}(\gamma), ..., H_{u_l}^{k,N-1}(\gamma))^T,$$

such that

$$G_k((H_{u_l}^{k,1}(\gamma), ..., H_{u_l}^{k,p-1}(\gamma), \gamma, H_{u_l}^{k,p}(\gamma), ..., H_{u_l}^{k,N-1}(\gamma))^T) = 0$$

for all $\gamma \in \mathcal{M}_1$. Also we have

$$u_l = (H_{u_l}^{k,1}(u_l^p), ..., H_{u_l}^{k,p-1}(u_l^p), u_l^p, H_{u_l}^{k,p}(u_l^p), ..., H_{u_l}^{k,N-1}(u_l^p))^T.$$

Now let

$$\mathcal{M} = \{u \in U : (u^1, ...u^{p-1}, u^{p+1}, .., u^N) \in \mathcal{M}_2, u^p \in \mathcal{M}_1\}.$$

If there exists a state $u' \in \mathcal{M}$ such that $G_k(u') = 0$, then there exists a $\gamma \in \mathcal{M}_1$, such that

$$G_k(u') = G_k((H_{u_l}^{k,1}(\gamma), ..., H_{u_l}^{k,p-1}(\gamma), \gamma, H_{u_l}^{k,p}(\gamma), ..., H_{u_l}^{k,N-1}(\gamma))^T)$$

Per definition is $\mathcal{M}$ a neighborhood of u_l. Thus we can define

$$S^k_{u_l}(\xi) = \begin{pmatrix} H^{k,1}_{u_l}(\xi + u^p_l) \\ \vdots \\ H^{k,p-1}_{u_l}(\xi + u^p_l) \\ \varepsilon + u^p_l \\ H^{k,p}_{u_l}(\xi + u^p_l) \\ \vdots \\ H^{k,N-1}_{u_l}(\xi + u^p_l) \end{pmatrix}.$$

Then $S^k_{u_l}(\xi)$ is a function from $E' = \{\xi \in \mathbb{R} : \xi + u^p_l \in \mathcal{M}_1\}$ into $\mathcal{M}$. The set $\{S^k_{u_l}(\xi) : \xi \in E'\}$ consists of all states u in $\mathcal{M}$, such that

$$l_{k,u_l}(u)^T r_{j,u_l} = 0$$

and so

$$A_{u_l}(u)(u - u_l) = s(u - u_l),$$

where $s = \lambda_{k,u_l}(u)$. We can find such a function for each $k \in \{1, ..., N\}$. So we can choose a neighborhood E of 0, such that the states $u_r \in \{S^k_{u_l}(\xi) : \xi \in E, k \in \{1, ..., N\}\}$ are the only states in a neighborhood of u_l, for which a real number s exists with

$$s(u_r - u_l) = F(u_l) - F(u_r).$$

This finishes the proof. □

To conclude our findings let $u_l \in U$ be a fixed state, then in a neighborhood of u_l the states $u_r \in \{S^k_{u_l}(\xi) : \xi \in E, k \in \{1, ..., N\}\}$, are the only states, such that

$$u(x,t) = \begin{cases} u_l & \text{if } x < st \\ u_r & \text{if } x > st \end{cases} \tag{2.5}$$

is a weak solution of the Riemann problem for (1.1) with initial data

$$u_0(x) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0\ . \end{cases} \tag{2.6}$$

Also $s = \lambda_{k,u_l}(u_r)$ for some $k \in \{1, ..., N\}$. We can now define shock waves like in [Lax57].

Definition 2.16. *We call the discontinuity in the weak solution* (2.5) *of the initial value problem* (1.1), (2.6) *a k-shock wave,* $k \in \{1, ..., N\}$, *if following two conditions hold*

(i) $\lambda_k(u_r) < s < \lambda_{k+1}(u_r)$

(ii) $\lambda_{k-1}(u_l) < s < \lambda_k(u_l)$

These conditions are called the Lax entropy conditions.

Remark 2.17. If $N = 1$ then the conditions simplify to

$$F'(u_l) < s < F'(u_r)$$

If the discontinuity in the weak solution (2.5) is a k-shock wave, then we say the state u_r is connected to the state u_l on the right by a k-shock wave. Our aim is to find the set of all states in a neighborhood of u_l, that can be connected to the state u_l on the right by a k-shock wave. The states in a neighborhood of u_l, which can be connected to a u_l on the right by a k-shock wave are obviously a subset of $\{S^k_{u_l}(\xi) : \xi \in E, k \in \{1, ..., N\}\}$. But before do that, we will prove the following corollary of the theorem before. The proof is taken from [GR21] and [Lax57].

Corollary 2.18. *Let $S^k_{u_l}$, $k \in \{1, ..., N\}$, be the functions obtained in the theorem before, then we can choose parametrization, so that*

(i) $\frac{d}{d\varepsilon} S^k_{u_l}(0) = r_k(u_l)$

(ii) $\frac{d^2}{d\varepsilon^2} S^k_{u_l}(0) = \nabla r_k(u_l) \cdot r_k(u_l)$

(iii) $\frac{d}{d\varepsilon} \lambda_{k.u_l}(S^k_{u_l}(0)) = \frac{1}{2}$,

where ε is the new parameter.

Proof. Let $u_l \in U$ and $k \in \{1, ..., N\}$, then from the proof of the last theorem we know

$$0 = l_{j,u_l}(S^k_{u_l}(\xi))^T (S^k_{u_l}(\xi) - S^k_{u_l}(0))$$

for all $j \in \{1, ..., N\} \backslash \{k\}$. Thus we can write

$$0 = \lim_{\xi \to 0} l_{j,u_l}(S^k_{u_l}(\xi))^T (\frac{1}{\xi}(S^k_{u_l}(\xi) - S^k_{u_l}(0))) = l_j(u_l)^T \frac{d}{d\xi} S^k_{u_l}(0).$$

Since $l_j(u_l)^T r_k(u_l) = 0$ for all $j \in \{1, ..., N\} \backslash \{k\}$ we get

$$\frac{d}{d\xi} S^k_{u_l}(0) = \alpha r_k(u_l)$$

for some $\alpha \in \mathbb{R}$. We can choose a new parameter δ, such that $\xi = \frac{1}{\alpha}\delta$. So we obtain

$$\frac{d}{d\delta} S^k_{u_l}(0) = r_k(u_l)$$

Now if we differentiate the jump condition

$$\lambda_{k,u_l}(S^k_{u_l}(\delta))(S^k_{u_l}(\delta) - u_l) = F(S^k_{u_l}(\delta)) - F(u_l)$$

with respect to δ, we obtain

$$\frac{d}{d\delta}\lambda_{k,u_l}(S^k_{u_l}(\delta))(S^k_{u_l}(\delta) - u_l) + \lambda_{k,u_l}(S^k_{u_l}(\delta))\frac{d}{d\xi} S^k_{u_l}(\delta) = dF(S^k_{u_l}(\delta))\frac{d}{d\delta} S^k_{u_l}(\delta)$$

and

$$\begin{aligned} &\frac{d^2}{d\delta^2}\lambda_{k,u_l}(S^k_{u_l}(\delta))(S^k_{u_l}(\delta) - u_l) + 2\frac{d}{d\delta}\lambda_{k,u_l}(S^k_{u_l}(\delta))\frac{d}{d\delta} S^k_{u_l}(\delta) + \lambda_{k,u_l}(S^k_{u_l}(\delta))\frac{d^2}{d\delta^2} S^k_{u_l}(\delta) \\ &= \frac{d}{d\delta} dF(S^k_{u_l}(\delta))\frac{d}{d\delta} S^k_{u_l}(\delta) + dF(S^k_{u_l}(\delta))\frac{d^2}{d\delta^2} S^k_{u_l}(\delta) \end{aligned}$$

For $\delta = 0$ we get

$$(dF(u_l) - \lambda_k(u_l))\frac{d^2}{d\delta^2} S^k_{u_l}(0) + \frac{d}{d\delta} dF(S^k_{u_l}(0)) r_k(u_l) = 2\frac{d}{d\delta}\lambda_{k,u_l}(S^k_{u_l}(0)) r_k(u_l). \qquad (2.7)$$

If we differentiate

$$dF(S^k_{u_l}(\delta))r_k(S^k_{u_l}(\delta)) = \lambda_k(S^k_{u_l}(\delta))r_k(S^k_{u_l}(\delta))$$

with respect to δ we get

$$\begin{aligned}&\frac{d}{d\delta}dF(S^k_{u_l}(\delta))r_k(S^k_{u_l}(\delta)) + dF(S^k_{u_l}(\delta))\frac{d}{d\delta}r_k(S^k_{u_l}(\delta))\\ &= \nabla\lambda_k(S^k_{u_l}(\delta)) \cdot \frac{d}{d\delta}S^k_{u_l}(\delta)r_k(S^k_{u_l}(\delta)) + \lambda_k(S^k_{u_l}(\delta))\frac{d}{d\delta}r_k(S^k_{u_l}(\delta)).\end{aligned}$$

For $\delta = 0$ this becomes

$$(dF(u_l) - \lambda_k(u_l))\nabla r_k(u_l) \cdot r_k(u_l) + \frac{d}{d\delta}dF(S^k_{u_l}(0))r_k(u_l) - \nabla\lambda_k(u_l)r_k(u_l)r_k(u_l) = 0. \tag{2.8}$$

Now we can subtract (2.8) from (2.7) and obtain

$$\begin{aligned}(dF(u_l) - \lambda_k(u_l))(\frac{d^2}{d\delta^2}S^k_{u_l}(0) - \nabla r_k(u_l) \cdot r_k(u_l))&\\ + \nabla\lambda_k(u_l)r_k(u_l)r_k(u_l) = 2\frac{d}{d\delta}\lambda_{k,u_l}(S^k_{u_l}(0))r_k(u_l).&\end{aligned} \tag{2.9}$$

If we multiply (2.9) with $l_k(u_l)^T$ on the left we get

$$\frac{d}{d\delta}\lambda_{k,u_l}(S^k_{u_l}(0)) = \frac{1}{2}\nabla\lambda_k(u_l) \cdot r_k(u_l) = \frac{1}{2}.$$

We can use that in (2.9), so

$$(dF(u_l) - \lambda_k(u_l))(\frac{d^2}{d\delta^2}S^k_{u_l}(0) - \nabla r_k(u_l) \cdot r_k(u_l)) = 0.$$

So there exists a $\beta \in \mathbb{R}$, such that

$$\frac{d^2}{d\delta^2}S^k_{u_l}(0) = \nabla r_k(u_l) \cdot r_k(u_l) + \beta r_k(u_l)$$

Again we can choose a new parameter ε, so that

$$\delta = \varepsilon - \frac{1}{2}\beta\varepsilon^2.$$

With that we obtain

$$\frac{d^2}{d\varepsilon^2}S^k_{u_l}(0) = \nabla r_k(u_l) \cdot r_k(u_k).$$

Also with this parametrization we have

$$\frac{d}{d\varepsilon}S^k_{u_l}(0) = r_k(u_l)$$

and

$$\frac{d}{d\varepsilon}\lambda_{k.u_l}(S^k_{u_l}(0)) = \frac{1}{2}.$$

This finishes the proof. □

Henceforth we choose the parametrization of $S^k_{u_l}$ in such way, that the condition (i) - (iii) from the corollary hold. We now can prove the main theorem of this subsection. The theorem and the proof are based on [Smo94].

Theorem 2.19. *Let $u_l \in U$ and $S^k_{u_l}(\varepsilon)$, $k \in \{1, ..., N\}$, the functions obtained in the last theorem. Then if ε is sufficiently small, the state $S^k_{u_l}(\varepsilon)$ can be connected to u_l on the right by a k-shock wave, if and only if $\varepsilon < 0$.*

Proof. Choose $u_l \in U$. We know for some small ε the state $S^k_{u_l}(\varepsilon)$, $k \in \{1, ..., N\}$, can be connected to $u_l = S^k_{u_l}(0)$ on the right by a discontinuity wave. The discontinuity is a k-shock wave if

(i) $\lambda_k(S^k_{u_l}(\varepsilon)) < \lambda_{k,u_l}(S^k_{u_l}(\varepsilon)) < \lambda_{k+1}(S^k_{u_l}(\varepsilon))$

(ii) $\lambda_{k-1}(S^k_{u_l}(0)) < \lambda_{k,u_l}(S^k_{u_l}(\varepsilon)) < \lambda_k(S^k_{u_l}(0))$

Note that

$$\frac{d}{d\varepsilon}(\lambda_k(S^k_{u_l}(0)) - \lambda_{k,u_l}(S^k_{u_l}(0))) = \nabla\lambda_k(u_l) \cdot r_k(u_l) - \frac{d}{d\varepsilon}\lambda_{k,u_l}(S^k_{u_l}(0)) = 1 - \frac{1}{2} > 0. \tag{2.10}$$

Assume ε is small enough and $\varepsilon \geq 0$ then $\lambda_k(S^k_{u_l}(\varepsilon)) \geq \lambda_{k,u_l}(S^k_{u_l}(\varepsilon))$. So the discontinuity is not a k-shock wave. Now assume $\varepsilon < 0$ sufficiently small, then $\lambda_k(S^k_{u_l}(\varepsilon)) < \lambda_{k,u_l}(S^k_{u_l}(\varepsilon))$ and since

$$\lim_{\varepsilon \to 0} \lambda_{k,u_l}(S^k_{u_l}(\varepsilon)) = \lambda_k(u_l),$$
$$\lim_{\varepsilon \to 0} \lambda_{k+1}(S^k_{u_l}(\varepsilon)) = \lambda_{k+1}(u_l),$$

we get $\lambda_{k,u_l}(S^k_{u_l}(\varepsilon)) < \lambda_{k+1}(S^k_{u_l}(\varepsilon))$, because $\lambda_k(S^k_{u_l}(0)) < \lambda_{k+1}(S^k_{u_l}(0))$. So we have

$$\lambda_k(S^k_{u_l}(\varepsilon)) < \lambda_{k,u_l}(S^k_{u_l}(\varepsilon)) < \lambda_{k+1}(S^k_{u_l}(\varepsilon))$$

for $\varepsilon < 0$ sufficiently small. Also from (2.10) we get $\lambda_{k,u_l}(S^k_{u_l}(\varepsilon)) < \lambda_k(S^k_{u_l}(0))$ and since

$$\lim_{\varepsilon \to 0} \lambda_{k,u_l}(S^k_{u_l}(\varepsilon)) = \lambda_k(S^k_{u_l}(0)) > \lambda_{k-1}(S^k_{u_l}(0))$$

we have

$$\lambda_{k-1}(S^k_{u_l}(0)) < \lambda_{k,u_l}(S^k_{u_l}(\varepsilon)) < \lambda_k(S^k_{u_l}(0))$$

for $\varepsilon < 0$ sufficiently small. ☐

So it follows, if $u_l \in U$, then there exists an interval $[\varepsilon_0, 0)$ with $\varepsilon_0 < 0$ sufficiently small, such that $\{S^k_{u_l}(\varepsilon) : \varepsilon \in [\varepsilon_0, 0)\}$ consists of all states in a neigborhood of u_l, that can be connected to u_l on the right by a k-shock wave.

2.4 Solution of the Riemann problem

Now we can finally prove the existence of weak solutions of the Riemann problem for (1.1). Let $u_l \in U$ and $k \in \{1, ..., N\}$. According to the previous two sections, we know that there are constants $\gamma_0 > 0$ and $\varepsilon_0 < 0$ such that in a neighborhood of u_l the set $\{R^k_{u_l}(\varepsilon) : \varepsilon \in [0, \gamma_0]\}$ consists of all states in that neighborhood that can be connected to u_l on the right by a k-centered simple wave and $\{S^k_{u_l}(\varepsilon) : \varepsilon \in [\varepsilon_0, 0)\}$ consists of states in that neighborhood, that can be connected to u_l on the right by a k-shock wave. Similarly to [GR21], we can define a function $U^k_{u_l} : [\varepsilon_0, \gamma_0] \to U$ by

$$U^k_{u_l}(\varepsilon) = \begin{cases} S^k_{u_l}(\varepsilon) & \text{if } \varepsilon < 0 \\ R^k_{u_l}(\varepsilon) & \text{if } \varepsilon \geq 0. \end{cases}$$

So the set $\{U^k_{u_l}(\varepsilon) : \varepsilon \in [\varepsilon_0, \gamma_0]\}$ consists of all states in a neighborhood of u_l, that can be connected to u_l on the right by a k-centered simple wave or a k-shock wave.

Lemma 2.20. *The function $U^k_{u_l}(\varepsilon)$ is twice continuously differentiable.*

Proof. We know the functions $S^k_{u_l}(\varepsilon)$ and $R^k_{u_l}(\varepsilon)$ are smooth. Also

$$\frac{d}{d\varepsilon}R^k_{u_l}(0) = r_k(u_l) = \frac{d}{d\varepsilon}S^k_{u_l}(0)$$

and

$$\frac{d^2}{d\varepsilon^2}R^k_{u_l}(0) = \nabla r_k(u_l) \cdot r_k(u_l) = \frac{d^2}{d\varepsilon^2}S^k_{u_l}(0).$$

So $U^k_{u_l}(\varepsilon)$ is twice continuously differentiable in $\varepsilon = 0$. □

Remark 2.21. Since $U^k_{u_l}(\varepsilon)$ is twice continuously differentiable in $\varepsilon = 0$ we get

$$U^k_{u_l}(\varepsilon) = u_l + \varepsilon r_k(u_l) + \frac{1}{2}\varepsilon^2 \nabla r_k(u_l) \cdot r_k(u_l) + \mathcal{O}(\varepsilon^3). \tag{2.11}$$

Assume $u_0, u_1, u_2 \in U$, such that $u_1 = U^k_{u_0}(\varepsilon_1)$ and $u_2 = U^{k+1}_{u_1}(\varepsilon_2)$ for some $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ and $k \in \{1, ..., N-1\}$. Then u_0 can be connected to u_1 on the right by a k-centered simple wave or a k-shock wave and u_1 can be connected to u_2 on the right by a $(k+1)$-centered simple wave or a $(k+1)$-shock wave. Thus the Riemann problems for (1.1) with initial data

$$u_0^1(x) = \begin{cases} u_0 & \text{if } x < 0 \\ u_1 & \text{if } x > 0 \end{cases} \tag{2.12}$$

and

$$u_0^2(x) = \begin{cases} u_1 & \text{if } x < 0 \\ u_2 & \text{if } x > 0 \end{cases} \tag{2.13}$$

have weak solutions. We can put these solutions together to obtain a solution for the Riemann problem for (1.1) with initial data

$$u_0(x) = \begin{cases} u_0 & \text{if } x < 0 \\ u_2 & \text{if } x > 0. \end{cases} \tag{2.14}$$

Assume for the moment, that $\varepsilon_1 \geq 0$ and $\varepsilon_2 < 0$, than the Riemann problem for (1.1) with initial data (2.12) has the solution

$$v_1(x,t) = \begin{cases} u_0 & \text{if } \frac{x}{t} < \lambda_k(u_0) \\ R^k_{u_0}(\frac{x}{t}) & \text{if } \lambda_k(u_0) \leq \frac{x}{t} \leq \lambda_k(u_1) \\ u_1 & \text{if } \lambda_k(u_1) < \frac{x}{t} \end{cases}$$

for $t > 0$ and $u(x,0) = u_0^1(x)$ for $t = 0$. The Riemann problem for (1.1) with initial data (2.13) has the solution

$$v_2(x,t) = \begin{cases} u_1 & \text{if } x < \lambda_{(k+1),u_l}(u_2)t \\ u_2 & \text{if } \lambda_{(k+1),u_l}(u_2)t < x. \end{cases}$$

By saying we put these solution together we mean we define a function $u : \mathbb{R} \times \mathbb{R}_+ \to U$ by

$$u(x,t) = \begin{cases} u_0 & \text{if } \frac{x}{t} < \lambda_k(u_0) \\ R^1_{u_0}(\frac{x}{t}) & \text{if } \lambda_k(u_0) \leq \frac{x}{t} \leq \lambda_k(u_1) \\ u_1 & \text{if } \lambda_k(u_1) < \frac{x}{t} < \lambda_{(k+1),u_l}(u_2) \\ u_2 & \text{if } \lambda_{(k+1),u_l}(u_2) < \frac{x}{t} \end{cases}$$

for $t > 0$ and $u(x,0) = u_0(x)$ for $t = 0$. This is possible, because $\lambda_k(u_1) < \lambda_{(k+1),u_l}(u_2)$. The function u is a weak solution of the Riemann problem for (1.1) with initial data (2.14). Figure 5 shows u in the x-t-plane. The fan consisting of straight lines represents the k-centered simple wave and the single straight line the $(k+1)$-shock wave. We can also define such a weak solution u, if $\varepsilon_1, \varepsilon_2 \geq 0$, $\varepsilon_1, \varepsilon_2 < 0$ or $\varepsilon_1 < 0, \varepsilon_2 \geq 0$. With that knowledge we can prove the following existence theorem. The proof is based on [GR21] and [Eva98].

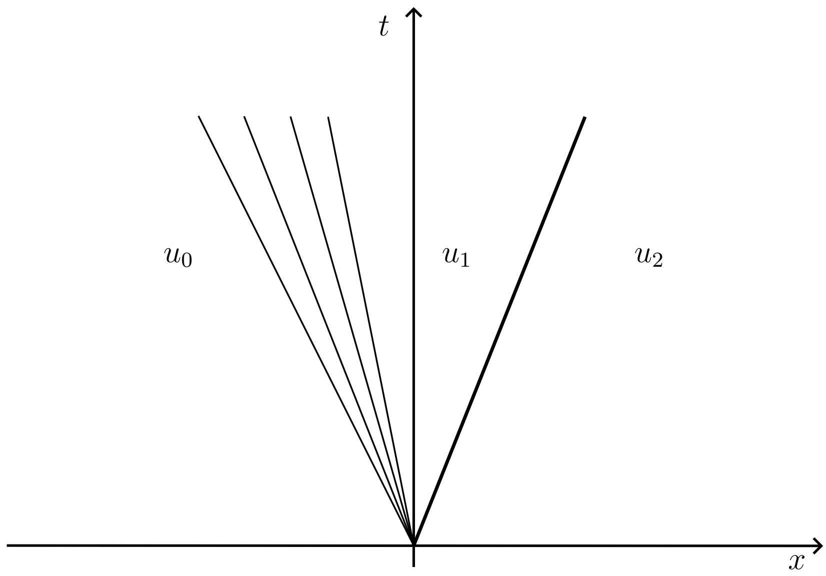

Figure 5: The weak solution u in the x-t-plane.

Theorem 2.22. *Let $u_l \in U$, then there exists a neighborhood $\mathcal{N}$ of u_l, such that if $u_r \in \mathcal{N}$, then the Riemann problem for (1.1) with initial data (2.1) has a weak solution. The solution consists of at most $(N-1)$ domains in the x-t-plane, where the solution is constant, that are connected by k-shock waves or k-centered simple waves, $k \in \{1, ..., N\}$. A solution of this kind is unique.*

Proof. Let $u_l \in U$ and let $\mathcal{E}$ be the set of all $\varepsilon = (\varepsilon_1, ..., \varepsilon_N)^T$ with $\varepsilon_1, ..., \varepsilon_N \in \mathbb{R}$ sufficiently small. Then we can define a mapping $\mathcal{T}_{u_l} : \mathcal{E} \to U$ by

$$\mathcal{T}_{u_l}(\varepsilon) = U^N(U^{N-1}(...(U^1(u_l;\varepsilon_1))...;\varepsilon_N),$$

where $U^k(u_l;\varepsilon_k) = U^k_{u_l}(\varepsilon_k)$, $k \in \{1, ..., N\}$. This mapping is twice continuously differentiable, because the functions $U^k(u_l;\varepsilon_k)$ are twice continuously differentiable. We also have

$$\mathcal{T}_{u_l}(0) = u_l.$$

We will show, that

$$\mathcal{T}_{u_l}(\varepsilon) = u_l + \sum_{k=1}^{N} \varepsilon_k r_k(u_l) + \mathcal{O}(|\varepsilon|^2) \tag{2.15}$$

With (2.11) we get

$$U^1_{u_l}(\varepsilon_1) = u_l + \varepsilon_1 r_1(u_l) + \mathcal{O}(\varepsilon_1^2)$$

Then we have

$$\begin{aligned} U^2((U^1(u_l;\varepsilon_1));\varepsilon_2) &= U^2(u_l + \varepsilon_1 r_1(u_l) + \mathcal{O}(\varepsilon_1^2);\varepsilon_2) \\ &= u_l + \varepsilon_1 r_1(u_l) + \mathcal{O}(\varepsilon_1^2) + \varepsilon_2 r_2(u_l + \varepsilon_k r_1(u_l) + \mathcal{O}(\varepsilon_1^2)) + \mathcal{O}(\varepsilon_2^2) \\ &= u_l + \varepsilon_1 r_1(u_l) + \varepsilon_2 r_2(u_l) + \mathcal{O}(\varepsilon_1^2 + \varepsilon_2^2). \end{aligned}$$

So by induction we obtain (2.15). Let $d\mathcal{T}_{u_l}$ be the Jacobian matrix of the mapping, then we obtain

$$d\mathcal{T}_{u_l}(u_l) = (r_1(u_l), ..., r_N(u_l)).$$

Since $r_1(u_l), ..., r_N(u_l)$ are linearly independent, $d\mathcal{T}_{u_l}(u_l)$ is invertible. So by the inverse function theorem [For17] there exists a neighborhood $\mathcal{N}$ of u_l such that if $u_r \in \mathcal{N}$ the equation

$$\mathcal{T}_{u_l}(\varepsilon) = u_r \tag{2.16}$$

has a unique solution $\varepsilon' \in \mathcal{E}$. Set $u_q = U^q(U^{q-1}(...(U^1(u_l;\varepsilon_1'))...;\varepsilon_q')$ for $q \in \{1, ..., N\}$ and $u_0 = u_l$. Because (2.16) has a solution, u_0 can be connected to u_1 on the right by a 1-shock wave or a 1-centered simple wave, u_1 can be connected to u_2 on the right by a 2-shock wave or a 2-centered simple wave and so on. Finally u_{N-1} can be connected to $u_N = u_r$ on the right by a N-shock wave or a N-centered simple wave. Thus for all $p \in \{0, ..., (N-1)\}$ the Riemann problems for (1.1) with initial data

$$u_0^p(x) = \begin{cases} u_p & \text{if } x < 0 \\ u_{p+1} & \text{if } x > 0 \end{cases}$$

have a weak solution

$$v_p(x,t) = \begin{cases} u_p & \text{if } \frac{x}{t} < \lambda_{p+1}(u_p) \\ R^{p+1}_{u_p}(\frac{x}{t}) & \text{if } \lambda_{p+1}(u_p) \leq \frac{x}{t} \leq \lambda_{p+1}(u_{p+1}) \\ u_{p+1} & \text{if } \lambda_{p+1}(u_{p+1}) < \frac{x}{t} \end{cases}$$

for $t > 0$ and $u(x,0) = u_0^p(x)$ for $t = 0$ if $\varepsilon_{p+1} \geq 0$ or

$$v_p(x,t) = \begin{cases} u_p & \text{if } x < \lambda_{p+1,u_l}(u_{p+1})t \\ u_{p+1} & \text{if } \lambda_{p+1,u_l}(u_{p+1})t < x \end{cases}$$

if $\varepsilon_{p+1} < 0$. We can put all these weak solutions together, like shown before, to get a weak solution of the Riemann problem for (1.1) with initial data

$$u_0(x) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0. \end{cases}$$

This weak solution consists of at most $(N-1)$ domains in the x-t-plane, where the solution is constant, which are connected by shock-waves or centered simple waves. Since the solution of (2.16) is unique, a solution of this kind is also unique. □

3 Glimm's Scheme

In the previous chapter the existence of weak solutions for the Riemann problem for (1.1) under certain conditions was shown. As we know, the Riemann problem is a special initial value problem. In this chapter we intent to present Glimm's scheme, which is also known as the random choice method. This scheme provided the earliest way to construct weak solutions of the initial value problem (1.1), (1.2) under specific conditions for the initial data. In other word, we will present a way to show, that the initial value problem (1.1), (1.2) has a weak solution if the initial data satisfies certain conditions. This chapter is based on [Smo94] with additions from [Daf16] and the original paper of James Glimm [Gli65].

3.1 Definition of the Scheme

Our aim is to find weak solutions of the initial value problem (1.1), (1.2). We assume without loss of generality, that U is a neighborhood of $0_N = (0, .., 0)^T \in \mathbb{R}^N$. Also we suppose (1.1) is hyperbolic and genuinely nonlinear in each characteristic field. For $u \in U$ we denote the Jacobian matrix of $F(u)$ by $dF(u)$. Like in the chapter before, let $\lambda_1(u) < ... < \lambda_N(u)$ be the eigenvalues of $dF(u)$ and $r_1(u), ..., r_N(u)$ and $l_1(u), ..., l_N(u)$ the normalized right and left eigenvectors to the corresponding eigenvalue.
We will now present Glimm's scheme, which can be used to construct weak solutions of the initial value problem (1.1),(1.2). The general idea of the scheme is to separate our initial value problem (1.1), (1.2) into Riemann problems. We know from chapter 2, that the Riemann problem has a weak solution, if the two states of the initial data are sufficiently close. The ideas is to obtain a weak solution of the initial value problem (1.1), (1.2) with help of the weak solutions of these Riemann problems.
We will define the scheme like [Smo94]. Choose neighborhoods $U_3 \subset U_2 \subset U_1 \subset U$ of 0_N. Let U_1 be a compact neighborhood and choose U_3 in such way, that by theorem 2.22 a weak solution of the Riemann problem for (1.1) with initial data (2.1) exists for every $u_l, u_r \in U_3$ and the range of the weak solution is in U_2. Remember this solution consists of at most $(N-1)$ domains in the x-t-plane, where the solution is constant. These domains are connected by shock-waves or centered simple waves. Also there is no other solution of this kind. We then choose $\Delta x, \Delta t > 0$, such that

$$\sup\{\lambda_j(u) : u \in U_2, j \in \{1, ..., N\}\} < \frac{\Delta x}{\Delta t}. \tag{3.1}$$

Thus $\frac{\Delta x}{\Delta t} = c$ for some $c > 0$ and henceforth we consider $\Delta t = \frac{1}{c}\Delta x$ as a function of Δx. Now define

$$\mathcal{Y} = \{(m, n) : (m, n) \in \mathbb{Z}^2, (m + n) \bmod 2 = 0, n \geq 0\}$$
$$\mathcal{A} = \prod_{(m,n)\in\mathcal{Y}} \{[(m-1)\Delta x, (m+1)\Delta x] \times \{n\Delta t\}\}.$$

We consider every factor of $\mathcal{A}$ as a probability space under $\frac{1}{2\Delta x}$ times the Lebesgue measure. So we consider $\mathcal{A}$ as the product space under the product measure. We denote the measure of the factor $[(m-1)\Delta x, (m+1)\Delta x] \times \{n\Delta t\}$ by $d\theta_{m,n}$ and the product measure by $d\theta$. Then we have

$$\int_{\mathcal{A}} d\theta = 1$$

because of the way we defined the measure. Now choose $\theta \in \mathcal{A}$ and let

$$\theta_{m,n} \in [(m-1)\Delta x, (m+1)\Delta x] \times n\Delta$$

be the components of θ. We call the points $\theta_{m,n}$ the mesh points. Since $\theta \in \mathcal{A}$ is arbitrary chosen, the scheme is also called the random choice method.
Let $u_{\Delta x,\theta}$ be an approximate solution for some $\Delta x > 0$ and $\theta \in \mathcal{A}$. Assume $u_{\Delta x,\theta}$ is defined at mesh points $\theta_{m-1,n}$, $\theta_{m+1,n}$, $((m-1), n), ((m+1), n) \in \mathcal{Y}$, and $u_{\Delta x,\theta}(\theta_{m-1,n}), u_{\Delta x,\theta}(\theta_{m+1,n}) \in U_3$. Then the initial value problem for (1.1) with initial data

$$v_0 = \begin{cases} u_{\Delta x,\theta}(\theta_{m-1,n}) & \text{if } x < m\Delta x \\ u_{\Delta x,\theta}(\theta_{m+1,n}) & \text{if } m\Delta x < x \end{cases}$$

can be considered as a Riemann problem and so has a solution v for $t \geq n\Delta t$, that consists of at most $(N-1)$ domains in the x-t-plane, where the solution is constant, which are connected by shock-waves or centered simple waves. Now we set

$$u_{\Delta x,\theta}(\theta_{m,n+1}) = \lim_{t \to (n+1)\Delta t} v(\theta^1_{m,n+1}, t)$$

where $\theta_{m,n+1} = (\theta^1_{m,n+1}, \theta^2_{m,n+1})^T$. Also we set

$$u_{\Delta x,\theta}(x,t) = v(x,t) \quad \text{for } (m-1)\Delta x \leq x \leq (m+1)\Delta x, \quad n\Delta t \leq t < (n+1)\Delta t.$$

Note that $u_{\Delta x,\theta}(x,t)$ is a weak solution of the Riemann problem for (1.1) with initial data

$$u_0(x) = u_{\Delta x,\theta}(x, n\Delta t)$$

in this rectangle. We can repeat this process as long as the range of the solution of the Riemann problems stay in U_3. Assume for some $n \in \mathbb{N}$, $u_{\Delta x,\theta}(x,t)$ is defined for all $x \in \mathbb{R}$, $n\Delta t \leq t < (n+1)\Delta t$, then for all $((m-1), n) \in \mathcal{Y}$, $u_{\Delta x,\theta}(x,t)$ is a weak solution of the initial value problem for (1.1) with initial data

$$u_0(x) = u_{\Delta x,\theta}(x, n\Delta t)$$

in the rectangle defined by $(m-1)\Delta x \leq x \leq (m+1)\Delta x$, $n\Delta t \leq t < (n+1)\Delta t$. If x is near $(m-1)\Delta x$ or $(m+1)\Delta x$ we get with (3.1)

$$u_{\Delta x,\theta}(x,t) = u_{\Delta x,\theta}(\theta_{m-1,n})$$

or

$$u_{\Delta x,\theta}(x,t) = u_{\Delta x,\theta}(\theta_{m+1,n}).$$

So $u_{\Delta x,\theta}(x,t)$ is weak solution of the initial value problem (1.1) with initial data

$$u_0(x) = u_{\Delta x,\theta}(x, n\Delta t)$$

for all $x \in \mathbb{R}, n\Delta t \leq t < (n+1)\Delta t$. At the moment we need to solve two problems:

(i) The range of the solutions of the Riemann problems need to be in U_3, such that we can define the approximate solution for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.

(ii) For $\Delta x \to 0$ and so $\Delta t \to 0$ the approximate solution has to converges to a weak solution of the initial value problem (1.1), (1.2).

We will start by solving (i). Therefore we are going to deduce bounds for the approximate solution.

3.2 Bounds for the Solutions of the Riemann Problem

In this section we will show some estimates, that will help us to deduce the bounds for the approximate solution. Suppose we have a Riemann problem for (1.1) with initial data

$$u_0(x) = \begin{cases} u_r & \text{if } x < 0 \\ u_l & \text{if } x > 0 \end{cases}$$

with $u_l, u_r \in U$, then we will call it for simplicity the (u_l, u_r) Riemann problem. If u_l and u_r are sufficiently close, the (u_l, u_r) Riemann problem has a weak solution, that consists of at most $(N-1)$ domains in the x-t-plane, where the solution is constant, which are connected by shock-waves or centered simple waves according to theorem 2.22. If we talk about a solution of the (u_r, u_l) Riemann problem, we always mean such a weak solution. From the proof of theorem 2.22 we know a unique $\varepsilon = (\varepsilon_1, ..., \varepsilon_N)^T \in \mathbb{R}^N$ and unique states $u_0, ..., u_N \in U$ with $u_0 = u_l$ and $u_N = u_r$ exists, such that

$$\begin{aligned} u_1 &= U^1_{u_0}(\varepsilon_1) \\ u_2 &= U^2_{u_1}(\varepsilon_2) \\ &\vdots \\ u_N &= U^N_{u_{N-1}}(\varepsilon_N) \end{aligned}$$

We call the states $u_0, ..., u_N$ the intermediate states and $\varepsilon = (\varepsilon_1, ..., \varepsilon_N)$ the magnitude of the waves of the solution of the (u_l, u_r) Riemann problem. Also for simplicity, we will call the j-shock wave or j-centered simple wave connecting the domain, where $u(x,t) = u_{j-1}$ to the domain, where $u(x,t) = u_j$ the j-wave connecting u_{j-1} to u_j. For the moment, let us call the j-wave connecting u_{j-1} to u_j α, then the strength of α is denoted by $|\alpha|$ and defined by $|\alpha| = |\varepsilon_j|$.
Suppose $u_l, u_m, u_r \in U$ and sufficiently close to 0_N. Then the (u_l, u_r), (u_l, u_m) and (u_m, u_r) Riemann problems have a solution. We know want to investigate how the magnitude of the waves of the solution of the (u_l, u_m) and (u_m, u_r) Riemann problems influence the magnitude of the waves of the (u_l, u_r) Riemann problem. Therefore we are going to prove the following lemma. The proof is from [Smo94] with some minor additions from [Gli65].

Lemma 3.1. *Let $u_l, u_m, u_r \in U$ be three states that are sufficiently close to 0_N. If the (u_l, u_r), (u_l, u_m) and (u_m, u_r) Riemann problems have a solution, where respectively $\varepsilon = (\varepsilon_1, ..., \varepsilon_N)^T, \gamma = (\gamma_1, ..., \gamma_N)^T, \delta = (\delta_1, ..., \delta_N)^T \in \mathbb{R}^N$ be the magnitudes of the waves and $u_0, ...u_N \in U$,$u'_0, ..., u'_n \in U$, $u''_0, ..., u''_N \in U$ be the intermediate states of the solutions. Then we have*

$$\varepsilon_j = \gamma_j + \delta_j + \mathcal{O}(|\gamma||\delta|) \tag{3.2}$$

for all $j \in \{1, ..., N\}$.

Proof. Assume $u_l, u_m, u_r \in U$ and sufficiently close to 0_N, such that the (u_l, u_r), (u_l, u_m) and (u_m, u_r) Riemann problems have a solution, where respectively $\varepsilon = (\varepsilon_1, ..., \varepsilon_N)^T, \gamma = (\gamma_1, ..., \gamma_N)^T, \delta = (\delta_1, ..., \delta_N)^T \in \mathbb{R}^N$ are the magnitudes of the waves and $u_0, ..., u_N \in U$, $u'_0, ..., u'_n \in U$ and $u''_0, ..., u''_N \in U$ are the intermediate states of the solutions. From the previous chapter we know for $i \in \{2, ..., N\}$

$$u_i = u_{(i-1)} + \varepsilon_i r_i(u_{(i-1)}) + \frac{1}{2}\varepsilon_i^2 \nabla r_i(u_{(i-1)}) \cdot r_i(u_{(i-1)}) + \mathcal{O}(\varepsilon_i^3).$$

By induction we obtain

$$u_r = u_l + \sum_{j=1}^{N} \varepsilon_j r_j(u_l) + \sum_{1\le p\le q\le N} \varepsilon_p\varepsilon_q \nabla r_q(u_l)\cdot r_p(u_l)(1-\frac{1}{2}\delta_{pq}) + \mathcal{O}(|\varepsilon|^3),$$

where δ_{pq} is the Kronecker delta. So we get

$$u_r - u_l = \sum_{j=1}^{N} \varepsilon_j r_j(u_l) + \sum_{1\le p\le q\le N} \varepsilon_p\varepsilon_q \nabla r_q(u_l)\cdot r_p(u_l)(1-\frac{1}{2}\delta_{pq}) + \mathcal{O}(|\varepsilon|^3) \tag{3.3}$$

and similarly

$$u_r - u_m = \sum_{j=1}^{i} \delta_j r_j(u_m) + \sum_{1\le p\le q\le i} \delta_p\delta_q \nabla r_q(u_m)\cdot r_p(u_m)(1-\frac{1}{2}\delta_{pq}) + \mathcal{O}(|\delta|^3). \tag{3.4}$$

Also for $i \in \{2, ..., N\}$ we have

$$u_{(i-1)} = u_i - \gamma_i r_j(u_i) + \frac{1}{2}\gamma_i^2 \nabla r_i(u_i)\cdot r_i(u_i) + \mathcal{O}(\gamma_i^3).$$

So again by induction we get

$$u_l - u_m = \sum_{j=1}^{i} -\gamma_j r_i(u_m) + \sum_{1\le q\le p\le i} \gamma_p\gamma_q \nabla r_q(u_m)\cdot r_p(u_m)(1-\frac{1}{2}\delta_{pq}) + \mathcal{O}(|\gamma|^3). \tag{3.5}$$

We can use (3.4) and (3.5) to obtain

$$u_r - u_l = u_r - u_m + u_m - u_l = \sum_{j=1}^{i}(\delta_j - \gamma_j) r_j(u_m) + \mathcal{O}((|\gamma| + |\lambda|)^2). \tag{3.6}$$

Let u_m be fixed and consider ε as a function of γ and δ, then according to [Gli65] and [Smo94], this function is continuously differentiable and since $\varepsilon = 0$ if $\gamma = \delta = 0$ we get $\varepsilon = \mathcal{O}(|\gamma| + |\delta|)$ if the states are close enough to 0_N. Also for $i \in \{1, ..., N\}$

$$r_i(u_l) = r_i(u_m) - \sum_{j=1}^{i} \gamma_j \nabla r_i(u_m)\cdot r_j(u_m) + \mathcal{O}(|\gamma|^2), \tag{3.7}$$

so we obtain

$$u_r - u_l = \sum_{j=1}^{N} \varepsilon_j r_j(u_l) + \mathcal{O}((|\gamma| + |\delta|)^2) = \sum_{j=1}^{N} \varepsilon_j r_j(u_m) + \mathcal{O}((|\gamma| + |\delta|)^2).$$

If we compare this with (3.6) we get

$$\varepsilon_i = \gamma_i + \delta_i + \mathcal{O}((|\gamma| + |\delta|)^2).$$

Now we can use this and 3.7 in 3.3 to find

$$u_r - u_r = \sum_{j=1}^{N} \varepsilon_j r_j(u_m) + \sum_{1\le p\le q\le N} \varepsilon_p\varepsilon_q \nabla r_q(u_m)\cdot r_p(u_m)(1-\frac{1}{2}\delta_{pq})$$
$$- \sum_{p,q=1}^{N} \varepsilon_p\gamma_q \nabla r_q(u_m)\cdot r_p(u_m) + \mathcal{O}((|\gamma| + |\delta|)^3).$$

So finally if we compare this with (3.4) and (3.5) we get

$$\begin{aligned}\sum_{j=1}^{N}(\varepsilon_j-\gamma_j-\delta_j)r_i(u_m) &= \sum_{j=1}^{N}(-\frac{1}{2}(\gamma_j+\delta_j)^2+(\gamma_j+\delta_j)\gamma_j+\frac{1}{2}\delta_j^2-\frac{1}{2}\gamma_j^2))\nabla r_i(u_m)\cdot r_i(u_m)\\ &+\sum_{1\le p<q\le N}(\delta_p\delta_q-(\gamma_p+\delta_p)(\gamma_q+\delta_q)+(\gamma_q+\delta_q)\gamma_p)\nabla r_q(u_m)\cdot r_p(u_m)\\ &+\sum_{1\le q<p\le N}(-\gamma_p\gamma_q+(\gamma_q+\delta_q)\gamma_p)\nabla r_q(u_m)\cdot r_p(u_m)+\mathcal{O}((|\gamma|+|\lambda|)^3)\\ &=\sum_{1\le p<q\le N}\gamma_q\delta_p(\nabla r_p(u_m)\cdot r_q(u_m)-\nabla r_q(u_m)\cdot r_p(u_m))+\mathcal{O}((|\gamma|+|\delta|)^3).\end{aligned}$$

Then we have

$$\sum_{j=1}^{N}(\varepsilon_j-\gamma_j-\delta_j)r_i(u_m)=\mathcal{O}(|\gamma||\delta|)$$

and since $r_1(u_m),...,r_N(u_m)$ are linearly independent we get $\varepsilon_j=\gamma_j+\delta_j+\mathcal{O}(|\gamma||\delta|)$ for all $j\in\{1,...,N\}$, which proves the lemma. □

Definition 3.2. *Let $u_l,u_m,u_r\in U$ and assume the the (u_l,u_m) and (u_m,u_r) Riemann problems have a solution. Let $\gamma=(\gamma_1,...,\gamma_N)$ and $\delta=(\delta_1,...,\delta_N)^T$ be the magnitudes of the waves and $u'_0,...,u'_n\in U$, $u''_0,...,u''_N\in U$ the intermediate states of the solutions. Let $j,k\in\{1,...,N\}$. We call the j-wave connecting u'_{j-1} to u'_j and the k-wave connecting u''_{k-1} to u''_k approaching if one of the following conditions hold*

(i) $j>k$

(ii) $j=k$ *and* $\gamma_k<0$ *or* $\delta_k<0$

Let

$$\mathcal{D}(\gamma,\delta)=\sum|\gamma_j||\delta_k|, \tag{3.8}$$

where the sum is over all pairs of (γ_j,δ_k), for which the j-wave connecting u'_{j-1} to u'_j and the k-wave connecting u''_{k-1} to u''_k are approaching.

This definition is according to [Smo94]. In the next theorem we will improve the estimate, which we obtained in the lemma. Again the theorem and proof are based on [Smo94].

Theorem 3.3. *Let $u_l,u_m,u_r\in U$ be three states that are sufficiently close to 0_N. If the (u_l,u_r), (u_l,u_m) and (u_m,u_r) Riemann problems have a solution, where respectively $\varepsilon=(\varepsilon_1,...,\varepsilon_N)^T,\gamma=(\gamma_1,...,\gamma_N)^T,\delta=(\delta_1,...,\delta_N)^T\in\mathbb{R}^N$ are the magnitudes of the waves and $u_0,...,u_N\in U$, $u'_0,...,u'_n\in U$, $u''_0,...,u''_N\in U$ are the intermediate states of the solutions. Then we have*

$$\varepsilon_i=\gamma_i+\delta_i+\mathcal{D}(\gamma,\delta)\mathcal{O}(1) \tag{3.9}$$

Proof. Like in proof in the lemma before, we assume $u_l,u_m,u_r\in U$ and suppose the states are close enough to 0_N, such that the (u_l,u_r), (u_l,u_m) and (u_m,u_r) Riemann problems have a solution. Let $\varepsilon=(\varepsilon_1,...,\varepsilon_N)^T,\gamma=(\gamma_1,...,\gamma_N)^T,\delta=(\delta_1,...,\delta_N)^T\in\mathbb{R}^N$ be the magnitudes of the waves and $u_0,...,u_N,u'_0,...,u'_n,u''_0,...,u''_N\in U$ be the intermediate states of the solutions.
Assume $\mathcal{D}(\gamma,\delta)=0$. Let $k\in\{1,...,N\}$ be the largest index, such that $\gamma_k\neq0$, so $\gamma=(\gamma_1,...,\gamma_k,0,...,0)^T$, then per definition of $\mathcal{D}(\gamma,\delta)$ we have $\delta_1,...\delta_{k-1}=0$. If $\gamma_k<0$, then $\delta_k=0$. So we have $u'_k=u_m=u''_k$ and we can put the solutions of the (u_l,u_m) and (u_m,u_r) Riemann problems together and get a solution of the (u_l,u_r) Riemann problem. Since such a solution is unique we get $\varepsilon_i=\gamma_i$ for $i\le k$ and $\varepsilon_i=\delta_i$ for $i>k$. If $\gamma_k>0$, then $\delta\ge0$. In that case u'_{k-1} can be connected to u_m on the right by a k-centered simple wave and u_m can be

connected to u_k'' on the right by a k-centered simple wave. Hence we can connect u_{k-1}' to u_k'' by a k-centered simple wave. So we can put the solutions together and obtain the solution of the (u_l, u_r) Riemann problem, since such solutions are unique. So $\varepsilon_i = \gamma_i$ for $i < k$, $\varepsilon_i = \delta_i$ for $i > k$ and $\varepsilon_k = \gamma_k + \delta_k$. So to sum up we get

$$\varepsilon_i = \gamma_i + \delta_i + \mathcal{D}(\gamma, \delta)\mathcal{O}(1)$$

for $i \in \{1, ..., N\}$. This proves the theorem for the case $\mathcal{D}(\gamma, \delta) = 0$.
Now we will prove the theorem by induction. If $\delta = (0, ..., 0)^T$, then $\mathcal{D}(\gamma, \delta) = 0$ and so

$$\varepsilon_i = \gamma_i + \delta_i + \mathcal{D}(\gamma, \delta)\mathcal{O}(1)$$

for $i \in \{1, ..., N\}$. Let $p \in \{1, ..., .N\}$ and assume

$$\varepsilon_i = \gamma_i + \delta_i + \mathcal{D}(\gamma, \delta)\mathcal{O}(1)$$

for $i \in \{1, ..., N\}$ if $\delta = (\delta_1, ..., \delta_{p-1}, 0, ..., 0)^T$. Now suppose $\delta = (\delta_1, ..., \delta_p, 0, ..., 0)^T$. Set $\Delta = (\delta_1, ..., \delta_{p-1}, 0, ..., 0)^T$ and $\Delta_0 = (0, ..., 0, \delta_p, 0, ..., 0)^T$, such that $\Delta + \Delta_0 = \delta$.
Then the (u_m, u_{p-1}'') Riemann problem has a solution, where $\Delta = (\delta_1, ..., \delta_{p-1}, 0, ..., 0)^T$ is the magnitude of the waves. Consider the (u_l, u_{p-1}'') Riemann problem. If the states u_r, u_m, u_l are sufficiently close to 0_N, it has a solution. Let $\rho = (\rho_1, ..., \rho_N)$ be the magnitude of the waves and $v_0, ..., v_n$ the intermediate states of this solution. Define $\mu = (\mu_1, ..., \mu_N)^T$ by $\mu_i = \rho_i$ if the j-wave connecting u_{j-1}' to u_j' and the p-wave connecting u_{p-1}'' to u_p'' do not approach and $\mu_i = 0$ otherwise. Also define $\nu = (\nu_1, ..., \nu_N)^T$ by $\nu_i = 0$ if $\mu_i = \rho_i$ and $\nu_i = \rho_i$ otherwise. Of course $i \in \{1, ..., N\}$. Note that $\mu_i = 0$ if $i > p$ and $\nu_i = 0$ if $i < p$. So $\rho = \mu + \nu$. Also we define

$$\tilde{u}_m = \begin{cases} v_{p-1} & \text{if } \mu_p = 0 \\ v_p & \text{if } \mu_p \neq 0. \end{cases}$$

Because the (u_l, u_m),(u_m, u_{p-1}'') and (u_l, u_{p-1}'') Riemann problems have solutions, where γ, Δ and ρ are the magnitudes of the waves of the solution, we can use the induction hypothesis and obtain

$$\rho_i = \gamma_i + \Delta_i + \mathcal{D}(\gamma, \Delta)\mathcal{O}(1).$$

for $i \in \{1, ..., N\}$. Because $\delta = \Delta + \Delta_0$, $\mathcal{D}(\gamma, \Delta) \leq \mathcal{D}(\gamma, \delta)$, we get

$$\rho_i = \gamma_i + \Delta_i + \mathcal{D}(\gamma, \delta)\mathcal{O}(1)$$

or

$$\rho_i = \gamma_i + \delta_i - \delta_{ip}\delta_p + \mathcal{D}(\gamma, \delta)\mathcal{O}(1). \tag{3.10}$$

Now consider the $(\tilde{u}_m, u_r)$ Riemann problem, which again has a solution if u_l, u_m, u_r are close enough to 0_N, and let $\pi = (\pi_1, ..., \pi_N)^T$ be the magnitude of the waves of the solution. Since Δ_0 is the magnitude of the waves of the solution of the (u_{p-1}'', u_r) Riemann problem and $\nu = (\nu_1, ..., \nu_N)^T$ is the magnitude of the waves of the solution of the $(\tilde{u}_m, u_{p-1}'')$ Riemann problem, we get with the lemma before

$$\pi_i = \nu_i + \delta_{ip}\delta_p + |\nu||\delta_p|\mathcal{O}(1).$$

for $i \in \{1, ..., N\}$. If $i < p$, $\nu_i = 0$ so $|\nu_i||\delta_p| = 0$. If $i = p$ we either have $\nu_p = 0$ and so $|\nu_p||\delta_p| = 0$ or $\nu_p = \rho_p$, when the p-wave connecting u_{p-1}' and u_p' and the p-wave connecting u_{p-1}'' and u_p'' are approaching. With 3.10 we get $\nu_p = \gamma_p + \mathcal{D}(\gamma, \delta)\mathcal{O}(1)$ in that case. So this gives us

$$|\nu_p||\delta_p| \leq |\delta_p|(|\gamma_p| + \mathcal{D}(\gamma, \delta)\mathcal{O}(1)) \leq (1 + |\delta_p|\mathcal{O}(1))\mathcal{D}(\gamma, \delta) \leq \mathcal{D}(\gamma, \delta)\mathcal{O}(1)$$

since $|\gamma_p||\delta_p| \leq \mathcal{D}(\gamma, \delta)$. If $i > p$ then the i-wave connecting u'_{i-1} and u'_i and the p-wave connecting u''_{p-1} and u''_p are approaching and we get with 3.10 like before

$$|\nu_i||\delta_p| \leq \mathcal{D}(\gamma, \delta)\mathcal{O}(1)$$

since $\delta_i = 0$ for $i > p$. We can combine our findings to get $|\nu||\delta_p| \leq \mathcal{D}(\gamma, \delta)\mathcal{O}(1)$. And so

$$\pi_i = \nu_i + \delta_{ip}\delta_p + \mathcal{D}(\gamma, \delta)\mathcal{O}(1). \tag{3.11}$$

Now define $\tilde{\pi} = \nu + \Delta_0$. Then if the states u_l, u_m, u_r are close enough, there exists a state $\tilde{u}_r$, such that $\tilde{\pi} = (\tilde{\pi}_1, ..., \tilde{\pi}_N)^T$ is the magnitude of the waves of of the solution of the $(\tilde{u}_m, \tilde{u}_r)$ Riemann problem. Also $\mu = (\mu_1, ..., \mu_N)^T$ is the magnitude of the waves of the solution of the $(u_l, \tilde{u}_m)$ Riemann problem. Note

(i) if $i < p$: $\tilde{\pi}_i = 0$

(ii) if $i = p$: $\mu_i = 0$ or $\nu_p = 0$ and so the p-wave connecting u'_{j-p} to u'_p and the p-wave connecting u''_{p-1} to u''_p do not approach

(iii) if $i > p$: $\mu_i = 0$

So $\mathcal{D}(\mu, \tilde{\pi}) = 0$. Let us consider the $(u_l, \tilde{u}_r)$ Riemann problem, for which a solution exists if u_l, u_m, u_r are close to 0_N and let $\varepsilon' = (\varepsilon'_1, ..., \varepsilon'_N)^T$ be the magnitude of the solution, then

$$\varepsilon'_i = \mu_i + \tilde{\pi}_i = \mu_i + \nu_i + \delta_{ip}\delta_p$$

since the theorem holds for $\mathcal{D}(\mu, \tilde{\pi}) = 0$. Also with $\rho = \nu + \mu$ and (3.10) we have

$$\gamma_i + \delta_i = \mu_i + \nu_i + \delta_{ip}\delta_p + \mathcal{D}(\gamma, \delta)\mathcal{O}(1).$$

So we obtain

$$\varepsilon'_i = \gamma_i + \delta_i + \mathcal{D}(\gamma, \delta)\mathcal{O}(1).$$

According to [Smo94] and [Gli65] there exists a continuously differentiable function for $\tilde{u}_m$, that maps the magnitudes of the waves of solutions of $(u_a, \tilde{u}_m)$ and $(\tilde{u}_m, u_b)$ Riemann problems to the magnitude of the waves of the (u_a, u_b) Riemann problem, where $u_a, u_b \in U$ and sufficiently close to $\tilde{u}_m$. Therefore we get

$$|\varepsilon - \varepsilon'| \leq |\pi - \tilde{\pi}|\mathcal{O}(1).$$

With $\tilde{\pi} = \nu_i + \delta_{ip}\delta_p$ and (3.11) we get $|\pi - \tilde{\pi}| = \mathcal{D}(\gamma, \delta)\mathcal{O}(1)$ and so we obtain

$$\varepsilon_i = \gamma_i + \delta_i + \mathcal{D}(\gamma, \delta)\mathcal{O}(1).$$

So the theorem holds for $\delta = (\delta_1, ..., \delta_p, 0, ..., 0)^T$. This completes the proof. □

3.3 Bounds on the Approximate Solution

In this section we want to obtain the desired bounds on the approximate solution. In [Smo94] this is a bit vague. So we will give additional details from [Daf16] and [Gli65]. Therefore we are going to define mesh curves, like in [Daf16]. Let $u_{\Delta x,\theta}$ be an approximative solution, then remember our mesh points are $\theta_{m,n}$, $(m, n) \in \mathcal{Y}$. We will now consider piecewise linear curves, which are connecting the mesh points. Let $\theta_{m_1,n_1}, ..., \theta_{m_r,n_r}$, $r \in \mathbb{N}$, be a finite sequence of mesh points, such that $m_{i+1} = m_i + 1$ and $n_{i+1} = n_i + 1$ or $n_{i+1} = n_i - 1$ for all $i \in \{1, ..., r - 1\}$. We connected θ_{m_i,n_i} to $\theta_{m_{i+1},n_{i+1}}$ by a linear curve. So we obtain a piecewise linear curve.

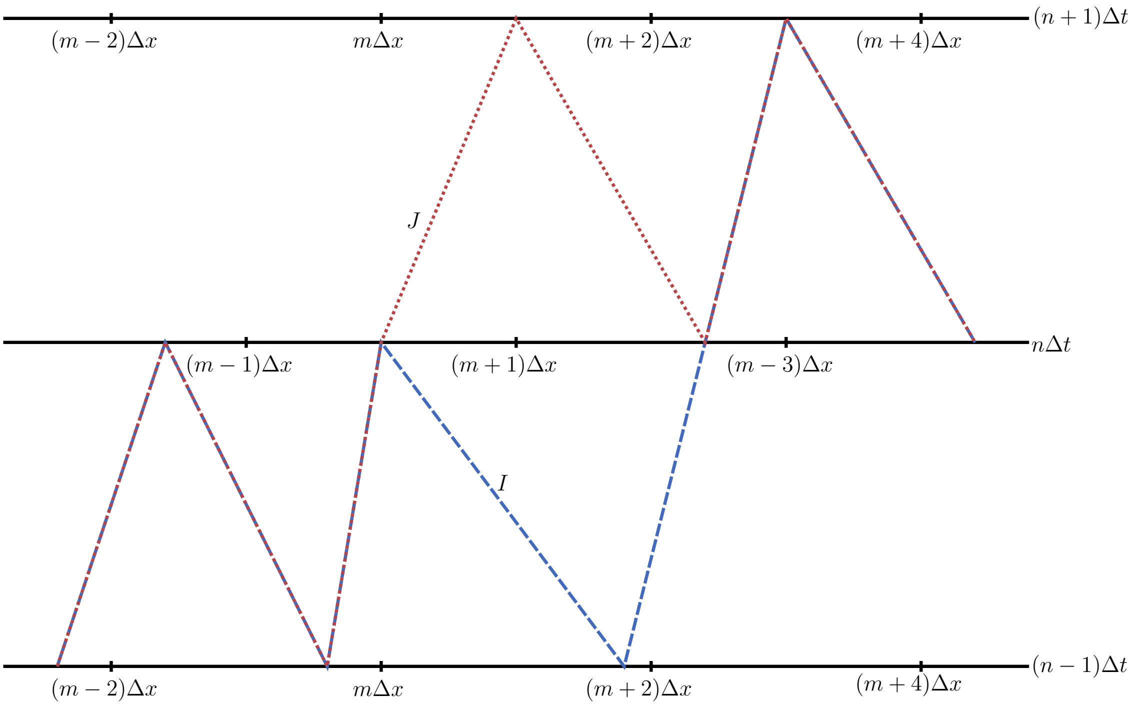

Figure 6: A mesh curve I with and immediate successor J.

We call such a curve a mesh curve and θ_{m_1,n_1} the start point and θ_{m_r,n_r} the end point of the mesh curve. Let us assume we have two mesh curves I and J, which have the same start and end points. We call J an immediate successor of I, if J and I go through the same mesh point except for one. As one can see in Figure 6, I and J enclose a region in the x-t-plane. We call such a region a diamond. If two mesh curves I and J have the same endpoints and there is a finite sequence $I = I_0, ..., I_r = J$, $r \in \mathbb{N}$, of mesh curves, where I_i is an immediate successor of I_{i-1} for all $i \in \{1, ..., r\}$ then we call J a successor of I. For simplicity if we say $\alpha \in I$, then α is a j-wave of the approximate solution crossing I, $j \in \{1, ..., N\}$. Also for simplicity we will just talk about waves. By that we mean j-waves for $j \in \{1, ..., N\}$. We will now expand our definition of approaching waves to mesh curves.

Definition 3.4. *Let J be a mesh curve and α a j-wave and a β k-wave of an approximative solution crossing J. We say α and β are approaching if one of the following conditions hold*

(i) $j \neq k$ and the wave with the higher index crosses J on the left of where the wave with the lower index crosses J

(ii) $j = k$ and $\alpha \neq \beta$ and at least one of the waves is a shock wave

Definition 3.5. *For a approximative solution and a mesh curve I we define*

(i) $\mathcal{L}(I) = \sum |\alpha|$, where the sum is over all waves α crossing I

(ii) $\mathcal{Q}(I) = \sum |\alpha||\beta|$, where the sum is over all waves α and β, which crosses I and approach

For simplicity we write

(i) $\mathcal{L}(I) = \sum\{|\alpha| : \alpha \in I\}$

(ii) $\mathcal{Q}(I) = \sum\{|\alpha||\beta| : \alpha, \beta \in I$ and approach$\}$

The definitions are from [Daf16]. We will use $\mathcal{L}$ and $\mathcal{Q}$ to prove bounds on the approximative solution. We start with a simple lemma.

Lemma 3.6. *Let I be a mesh curve, then $\mathcal{Q}(I) \leq \mathcal{L}(I)^2$.*

Proof. We have

$$\begin{aligned}\mathcal{Q}(I) &= \sum |\alpha||\beta| : \alpha, \beta \in I \text{ and approach}\} \\ &\leq \sum |\alpha||\beta| : \alpha, \beta \in I\} \\ &\leq \mathcal{L}(I)^2\end{aligned}$$

□

Let $u_r, u_m, u_l \in U_3$ and $\gamma = (\gamma_1, ..., \gamma_N)^T$, $\delta = (\delta_1, ..., \delta_N)^T$ and $\varepsilon = (\varepsilon_1, ..., \varepsilon_N)^T$ are the magnitudes of the waves of the solutions of the (u_l, u_m), (u_m, u_r) and (u_l, u_r) Riemann problems. According to [Gli65], with theorem 3.3, we can choose a neighborhood of $U_4 \subseteq U_3$ of 0_N, such that a constant $k_0 > 0$ exists, so that

$$|\varepsilon_i| \leq |\gamma_i| + |\delta_i| + k_0 \mathcal{D}(\gamma, \delta) \tag{3.12}$$

for all $i \in \{1, ..., N\}$, if the range of the solution of the (u_l, u_m) and (u_m, u_r) Riemann problems are in U_4. With that we can prove the following lemma, which gives us some properties of $\mathcal{L}$ and $\mathcal{Q}$. The proof is based on [Smo94].

Lemma 3.7. *Let $u_{\Delta x,\theta}$ be an approximate solution and let I be a mesh curve and J and immediate successor of I. Let $\theta_{m,n+1}$ be the mesh point on J, that is not on I and $\theta_{m,n-1}$ the mesh point on I, that is not on J. Assume $u_{\Delta x,\theta}$ is defined on the mesh curves and the range of the approximative solution on the mesh curves is in U_4. Then the $(u_{\Delta x,\theta}(\theta_{m-1,n}), u_{\Delta x,\theta}(\theta_{m,n-1}))$ Riemann problem has a solution, where $\gamma = (\gamma_1, ..., \gamma_N)^T$ is the magnitude of the waves and the $(u_{\Delta x,\theta}(\theta_{m,n-1}), u_{\Delta x,\theta}(\theta_{m+1,n}))$ Riemann problem has a solution, where $\delta = (\delta_1, ..., \delta_N)^T$ is the magnitude of the waves. Then*

$$\mathcal{L}(J) \leq \mathcal{L}(I) + Nk_0\mathcal{D}(\gamma, \delta) \tag{3.13}$$

and

$$\mathcal{Q}(J) - \mathcal{Q}(I) \leq \mathcal{L}(I)Nk_0\mathcal{D}(\gamma, \delta) - \mathcal{D}(\gamma, \delta). \tag{3.14}$$

Proof. The mesh curves I and J form a diamond, as depicted in figure 7. There are waves crossing in and out of this diamond. The waves are represented by straight lines. Let $u_m = u_{\Delta x,\theta}(\theta_{m,n-1})$, $v_p = u_{\Delta x,\theta}(\theta_{m-1,n})$ and $v_q = u_{\Delta x,\theta}(\theta_{m+1,n})$. So $v_q, u_m, v_p \in U_4$. By the uniqueness of the solutions, the waves crossing inside the diamond are the waves of the solutions of the (v_q, u_m) and (u_m, v_q) Riemann problems. Let $\gamma = (\gamma_1, ..., \gamma_N)^T$ be the magnitude of the waves of the solution of the (v_q, u_m) Riemann problem and let $\delta = (\delta_1, ..., \delta_N)^T$ be the magnitude of the waves of the solution of the (u_m, v_q) Riemann problem. The range of these solutions is also in U_4. Also the (v_p, v_q) Riemann problem has a solution. The waves of the solution are the waves crossing out of the diamond. We denote the magnitude of the waves of the solution by $\varepsilon = (\varepsilon_1, ..., \varepsilon_N)^T$. We can use the (3.12) to get

$$\begin{aligned}\mathcal{L}(J) &= \mathcal{L}(I) - \sum_{i=1}^{N} |\gamma_i| - \sum_{i=1}^{N} |\delta_i| + \sum_{i=1}^{N} |\varepsilon_i| \\ &\leq \mathcal{L}(I) - \sum_{i=1}^{N} |\gamma_i| - \sum_{i=1}^{N} |\delta_i| + \sum_{i=1}^{N} |\gamma_i + \delta_i + k_0\mathcal{D}(\gamma, \delta)| \\ &\leq \mathcal{L}(I) + Nk_0\mathcal{D}(\gamma, \delta).\end{aligned}$$

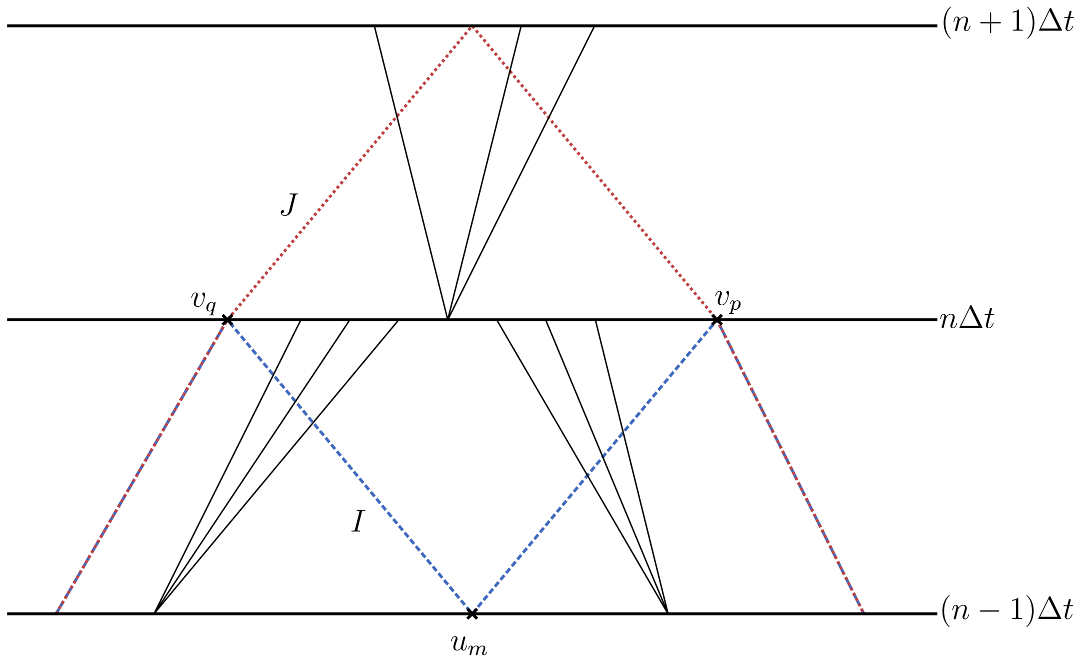

Figure 7: The diamond formed by I and J.

Now define $I_0 = J \cap I$ and I', J', such that $J = J' \cup I_0$ and $I = I' \cup I_0$. Also define

$$\mathcal{Q}(I_0, I') = \sum\{|\alpha||\beta| : \alpha \in I_0, \beta \in I' \text{ and approach}\}.$$

By $\alpha \in I_0$ we mean a wave α, which crosses I_0. Then

$$\begin{aligned}\mathcal{Q}(I) &= \mathcal{Q}(I_0) + \mathcal{Q}(I') + \mathcal{Q}(I_0, I')\\ \mathcal{Q}(J) &= \mathcal{Q}(I_0) + \mathcal{Q}(J') + \mathcal{Q}(I_0, J').\end{aligned}$$

We can see $\mathcal{Q}(J') = 0$ and $\mathcal{Q}(I') = \mathcal{D}(\gamma, \delta)$. So we get

$$\begin{aligned}\mathcal{Q}(I) &= \mathcal{Q}(I_0) + \mathcal{D}(\gamma, \delta) + \mathcal{Q}(I_0, I')\\ \mathcal{Q}(J) &= \mathcal{Q}(I_0) + \mathcal{Q}(I_0, J').\end{aligned}$$

For simplicity we call the j-waves, $j \in \{1, ..., N\}$, of the solutions of the (v_p, v_q), (v_p, u_m) and (u_m, v_q) Riemann problems just ε_j, γ_j and δ_j. Then

$$\mathcal{Q}(I_0, J') = \sum |\alpha||\varepsilon_j|,$$

where the sum is over all waves α that cross I_0 and waves ε_j, that approach. With (3.12) we get

$$\begin{aligned}\mathcal{Q}(I_0, J') &\leq \sum(|\alpha|(|\gamma_j| + |\delta_j| + k_0\mathcal{D}(\gamma, \delta)))\\ &\leq \sum(|\alpha|(|\gamma_j| + |\delta_j|)) + k_0 N \mathcal{L}(I_0)\mathcal{D}(\gamma, \delta)\end{aligned}$$

where the sum is over all waves α, that crosses I_0 and all j, for which α and ε_j approach. Now assume we have a k-wave α, $k \in \{1, .., N\}$, which crosses I_0 and approaches ε_j. If $k \neq j$ or α is a k-shock wave and $k = j$, then α approaches δ_j and γ_j. If $k = j$ and α is a k-centered simple wave then ε_j has to be a j-shock wave. In that case α approaches γ_j or δ_j if they are j-shock waves. Assume δ_j is a j-centered simple wave, so $\delta_j \geq 0$, then

$$|\varepsilon_j| \leq |\gamma_j + \delta_j + \mathcal{D}(\gamma, \delta)| \leq |\gamma_j + \mathcal{D}(\gamma, \delta)|.$$

If $\gamma_j \geq 0$ too, then

$$|\varepsilon_j| \leq \mathcal{D}(\gamma, \delta).$$

But so we get

$$\mathcal{Q}(I_0, J') \leq \mathcal{Q}(I_0, I') + k_0 N \mathcal{L}(I_0)\mathcal{D}(\gamma, \delta)$$

and so

$$\begin{aligned} \mathcal{Q}(J) - \mathcal{Q}(I) &= \mathcal{Q}(I_0, J') - \mathcal{Q}(I_0, I') - \mathcal{Q}(I') \\ &\leq k_0 N \mathcal{L}(I_0)\mathcal{D}(\gamma, \delta) - \mathcal{D}(\gamma, \delta) \\ &\leq k_0 N \mathcal{L}(I)\mathcal{D}(\gamma, \delta) - \mathcal{D}(\gamma, \delta). \end{aligned}$$

This proves the second estimate. □

Definition 3.8. *The Glimm functional $\mathcal{F}$ for a mesh curve I is defined by*

$$\mathcal{F}(I) = \mathcal{L}(I) + k\mathcal{Q}(I),$$

where $k \geq 2k_0 N$ is a constant.

This definition is from [Daf16]. With the Glimm functional we can prove the following theorem. The proof is again based on [Smo94].

Theorem 3.9. *Let I be a mesh curve with $k\mathcal{L}(I) \leq 1$ and J an successor of I. Assume that the approximative solution can be defined for all points on J and I and the range of the approximate solution is in U_4. Then*

$$\mathcal{F}(J) \leq \mathcal{F}(I)$$

and

$$\mathcal{L}(J) \leq 2\mathcal{L}(I).$$

Proof. Assume J is an immediate successor of I then with lemma 3.7 we get

$$\begin{aligned} \mathcal{F}(J) &= \mathcal{L}(J) + k\mathcal{Q}(J) \\ &\leq \mathcal{L}(I) + k_0 N \mathcal{D}(\gamma, \delta) + k(\mathcal{Q}(I) + k_0 N \mathcal{L}(I)\mathcal{D}(\gamma, \delta) - \mathcal{D}(\gamma, \delta)) \\ &= \mathcal{L}(I) + k\mathcal{Q}(I) + (k_0 N + k k_0 N \mathcal{L}(I) - k)\mathcal{D}(\gamma, \delta) \\ &= \mathcal{F}(I) + (k_0 N + k k_0 N \mathcal{L}(I) - k)\mathcal{D}(\gamma, \delta) \end{aligned}$$

where γ and δ are like in the lemma before. With

$$\begin{aligned} k_0 N + k k_0 N \mathcal{L}(I) - k &\leq k_0 N + k_0 N - k \\ &\leq 2k_0 N - 2k_0 N \\ &= 0 \end{aligned}$$

we get $\mathcal{F}(J) \leq \mathcal{F}(I)$. Let J now be an arbitrary successor of I and assume the approximate solution is defined on J and the range of the approximate solution is in U_4. From the definition of successors we know, there is a finite sequence $I_1, ..., I_M$, $M \in \mathbb{N}$, of immediate successors with $I_1 = I$ and $I_M = J$. So we get

$$\mathcal{F}(I) \geq \mathcal{F}(I_2) \geq ... \geq \mathcal{F}(J).$$

This proves the first estimate. With that we obtain

$$\begin{aligned}\mathcal{L}(J) &\leq \mathcal{L}(J) + k\mathcal{Q}(J) \\ &\leq \mathcal{L}(I) + k\mathcal{Q}(I) \\ &\leq \mathcal{L}(I) + k\mathcal{L}(I)^2 \\ &\leq \mathcal{L}(I) + \mathcal{L}(I) \\ &= 2\mathcal{L}(I).\end{aligned}$$

This proves the second estimate. □

Now let us define total variation according to [Ser99].

Definition 3.10. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and S an interval of $\mathbb{R}$, which can be either open or closed, then the total variation of f on S is*

$$T.V(f, S) = \sup_{\mathcal{P}} \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|$$

where $\mathcal{P}$ is the set of all finite increasing sequences $(x_0, ..., x_n) \subseteq S$.

A more general definition of total variation is given in [EG15].

Definition 3.11. *Let $\mathcal{V} \subseteq \mathbb{R}^p$ with $p \in \mathbb{N}$ and $g \in L^1(\mathcal{V})$, then*

$$T.V(g, \mathcal{V}) = \sup_{|\phi| \leq 1} \int_{\mathcal{V}} g(x) \text{div} \phi(x) dx$$

where $\phi \in C_0^1(\mathcal{V})^p$.

According to [Smo94] $\mathcal{L}(I)$ measurers the total variation of the approximate solution on a mesh curve I. We can use that to prove the main theorem of this section. The theorem and proof are from [Daf16].

Theorem 3.12. *Fix $0 \leq \tau_1 < \tau_2 < \infty$ and $-\infty < a \leq b < \infty$ and assume the approximative solution $u_{\Delta x,\theta}$ is defined for all $x \in \mathbb{R}$ and $t < n_c \Delta t$ and the range of $u_{\Delta x,\theta}$ is in U_4 for all $x \in \mathbb{R}$ and $t < n_c \Delta t$, where $n_c \in \mathbb{N}$ with $(n_c - 1)\Delta t \leq \tau_2 \leq n_c \Delta t$. If*

$$kT.V(u_{\Delta x,\theta}(\cdot, \tau_1), [a - c(\tau_2 - \tau_1) - 6\Delta x, b + c(\tau_2 - \tau_1) + 6\Delta x])$$

is sufficiently small, then

$$T.V(u_{\Delta x,\theta}(\cdot, \tau_2), [a, b]) \leq C_1 T.V(u_{\Delta x,\theta}(\cdot, \tau_1), [a - c(\tau_2 - \tau_1) - 6\Delta x, b + c(\tau_2 - \tau_1) + 6\Delta x])$$

where C_1 is a constant that only depends on F.
Let $x \in \mathbb{R}$ be a point of continuity of $u_{\Delta x,\theta}(\cdot, \tau_1)$ and $u_{\Delta x,\theta}(\cdot, \tau_2)$ and

$$kT.V(u_{\Delta x,\theta}(\cdot, \tau_1), [x - c(\tau_2 - \tau_1) - 6\Delta x, x + c(\tau_2 - \tau_1) + 6\Delta x])$$

be sufficiently small, then

$$|u_{\Delta x,\theta}(x, \tau_2) - u_{\Delta x,\theta}(x, \tau_1)| \leq C_2 T.V(u_{\Delta x,\theta}(\cdot, \tau_1), [x - c(\tau_2 - \tau_1) - 6\Delta x, x + c(\tau_2 - \tau_1) + 6\Delta x])$$

where C_2 is a constant, which also only depends on F.

Proof. We choose $n_1, n_2 \in \mathbb{N}$, such that $n_1\Delta t \leq \tau_1 \leq (n_1+1)\Delta t$ and $n_2\Delta t \leq \tau_2 \leq (n_2+1)\Delta t$. Then the approximative solution is defined for all $t < (n_2+1)\Delta t$. Also choose $m_1, m_2 \in \mathbb{N}$, such that there exists mesh points

$$\theta_{m_1+1,n_2+1} = (\theta^1_{m_1+1,n_2+1}, \theta^2_{m_1+1,n_2+1})$$

and

$$\theta_{m_1+3,n_2+1} = (\theta^1_{m_1+3,n_2+1}, \theta^2_{m_1+3,n_2+1})$$

with $\theta^1_{m_1+1,n_2+1} < a \leq \theta^1_{m_1+3,n_2+1}$ and mesh points

$$\theta_{m_2-3,n_2+1} = (\theta^1_{m_2-3,n_2+1}, \theta^2_{m_2-3,n_2+1})$$

and

$$\theta_{m_2-1,n_2+1} = (\theta^1_{m_2-1,n_2+1}, \theta^2_{m_2-1,n_2+1})$$

with $\theta^1_{m_2-3,n_2+1} \leq a < \theta^1_{m_2-1,n_2+1}$. Also set $m_3 = m_1 - (n_2 - n_1)$ and $m_4 = m_2 + (n_2 - n_1)$. Then we construct two mesh curves I and J.

(i) I has the start point θ_{m_3,n_1} and the end point θ_{m_4,n_1} and only has mesh points that lay on the lines $t = n_1\Delta t$ and $t = (n_1+1)\Delta t$.

(ii) J has the start point θ_{m_3,n_1}. From there it goes to the mesh point θ_{m_3+1,n_1+1}, then to the mesh point θ_{m_3+1,n_1+1} and so on until it reaches the mesh point θ_{m_1,n_2}. From here J lays between the lines $t = n_2\Delta t$ and $t = (n_2+1)\Delta t$ until it reaches the mesh point θ_{m_2,n_2}. From here it goes to the mesh point θ_{m_2-1,n_2-1}, then to the mesh point θ_{m_2-2,n_2-2} and so on until it reaches the mesh point θ_{m_4,n_1}. This is the end point of J.

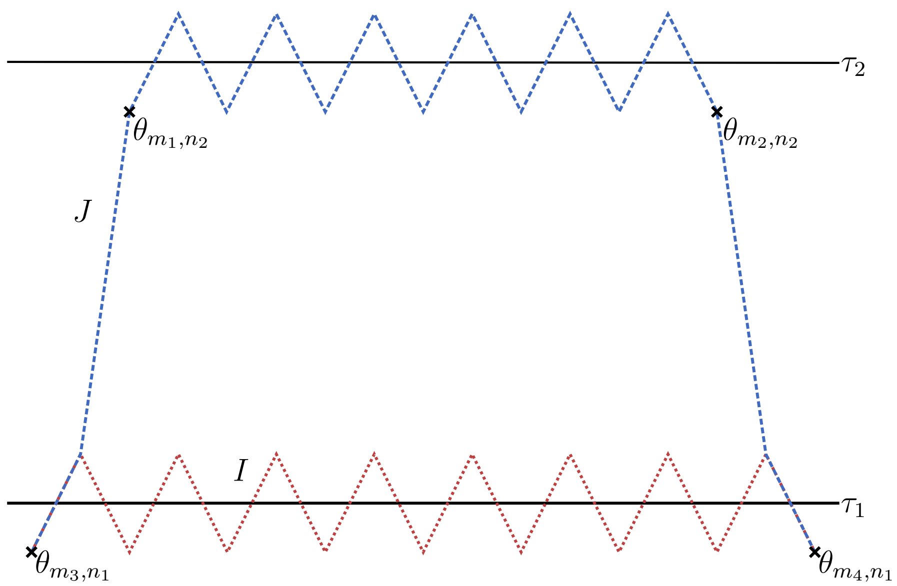

Figure 8: The mesh curves I and J.

For better understanding one may take a look at Figure 8. J is a successor of I and per definition of the scheme the approximative solution is defined at all point an J and I and the range of the approximative solution on I and J is in U_4. We assume

$$kT.V(u_{\Delta x,\theta}(\cdot,\tau_1), [a - c(\tau_2-\tau_1) - 6\Delta x, b + c(\tau_2-\tau_1) + 6\Delta x])$$

is sufficiently small. Since $\mathcal{L}(I)$ measures the total variation on I there exists a constant c_1 only dependent on F, such that

$$\mathcal{L}(I) \leq c_1 T.V(u_{\Delta x,\theta}(\cdot,\tau_1),[a-c(\tau_2-\tau_1)-6\Delta x, b+c(\tau_2-\tau_1)+6\Delta x]).$$

From the theorem before we know $\mathcal{L}(J) \leq 2\mathcal{L}(I)$ if $k\mathcal{L}(I) \leq 1$. This is given if

$$kT.V(u_{\Delta x,\theta}(\cdot,\tau_1),[a-c(\tau_2-\tau_1)-6\Delta x, b+c(\tau_2-\tau_1)+6\Delta x])$$

is sufficiently small. Also we have a constant c_2 only depending on F such that

$$T.V(u_{\Delta x,\theta}(\cdot,\tau_2),[a,b]) \leq c_2\mathcal{L}(J).$$

So we can combine these findings to obtain the first estimate with $C_1 = 2c_1c_2$.
Now let $x \in \mathbb{R}$ be a point of continuity of $u_{\Delta x,\theta}(\cdot,\tau_1)$ and $u_{\Delta x,\theta}(\cdot,\tau_2)$, then we choose $x = a = b$ and construct I and J like before. There is a point $(\tilde{x},\tilde{t})$ on I with $u_{\Delta x,\theta}(\tilde{x},\tilde{t}) = u_{\Delta x,\theta}(x,\tau_1)$ and a point (x',t') on J with $u_{\Delta x,\theta}(x',t') = u_{\Delta x,\theta}(x,\tau_2)$. Again assume

$$kT.V(u_{\Delta x,\theta}(\cdot,\tau_1),[a-c(\tau_2-\tau_1)-6\Delta x, b+c(\tau_2-\tau_1)+6\Delta x])$$

is sufficiently small. A constant c_3 only depending on F exists, such that

$$|u_{\Delta x,\theta}(x,\tau_2) - u_{\Delta x,\theta}(x,\tau_1)| \leq c_3(\mathcal{L}(I)+\mathcal{L}(J)).$$

Since

$$kT.V(u_{\Delta x,\theta}(\cdot,\tau_1),[a-c(\tau_2-\tau_1)-6\Delta x, b+c(\tau_2-\tau_1)+6\Delta x])$$

is sufficiently small, we get $k\mathcal{L}(I) \leq 1$ and so

$$\begin{aligned}|u_{\Delta x,\theta}(x,\tau_2) - u_{\Delta x,\theta}(x,\tau_1)| &\leq 3c_3\mathcal{L}(I)\\ &\leq 3c_3c_1T.V(u_{\Delta x,\theta}(\cdot,\tau_1),[x-c(\tau_2-\tau_1)-6\Delta x, x+c(\tau_2-\tau_1)+6\Delta x]).\end{aligned}$$

This proves the second estimate with $C_2 = 3c_1c_3$. □

Corollary 3.13. *Let $T.V(u_0,\mathbb{R})$ be sufficiently small and assume the approximate solution $u_{\Delta x,\theta}$ is defined for all $x \in \mathbb{R}$ and $t < n_2\Delta t$, where $n_2 \in \mathbb{N}$, and the range of the approximative solution is in U_4, then for any $0 \leq t < n_2\Delta t$*

$$||u_{\Delta x,\theta}(\cdot,t)||_\infty \leq ||u_0||_\infty + C_2T.V(u_0,\mathbb{R}).$$

Proof. Let $0 \leq t < n_2\Delta t$, then we can use the second estimate of the theorem with $\tau_1 = 0$ and $\tau_2 = t$ to obtain for every point of continuity $x \in \mathbb{R}$

$$\begin{aligned}|u_{\Delta x,\theta}(x,t) - u_{\Delta x,\theta}(x,0)| &\leq C_2T.V(u_{\Delta x,\theta}(\cdot,0),[x-c(\tau_2-\tau_1)-6\Delta x, x+c(\tau_2-\tau_1)+6\Delta x])\\ &\leq C_2T.V(u_{\Delta x,\theta}(\cdot,0),\mathbb{R})\\ &\leq C_2T.V(u_0,\mathbb{R}).\end{aligned}$$

Since C_2 depends only on F we get

$$\begin{aligned}||u_{\Delta x,\theta}(\cdot,t)||_\infty - ||u_{\Delta x,\theta}(\cdot,0)||_\infty &\leq ||u_{\Delta x,\theta}(\cdot,t) - u_{\Delta x,\theta}(\cdot,0)||_\infty\\ &\leq C_2T.V(u_0,\mathbb{R})\end{aligned}$$

and so

$$||u_{\Delta x,\theta}(\cdot,t)||_\infty \leq ||u_0||_\infty + C_2T.V(u_0,\mathbb{R}).$$

□

This corollary is based on [Daf16]. It ensures that we can define the approximative solution for all $t \geq 0$. We will show that in the following theorem.

Theorem 3.14. *There exists positive constants C_3 and C_4 sufficiently small, such that if*

(i) $||u_0||_\infty \leq C_3$

(ii) $T.V(u_0, \mathbb{R}) \leq C_4$

then the approximative solutions $u_{\Delta x,\theta}$ can be defined for all $t \geq 0$ and

(i) $||u_{\Delta x,\theta}(\cdot, t)||_\infty \leq ||u_0||_\infty + C_2 T.V(u_0, \mathbb{R})$

(ii) $T.V(u_{\Delta x,\theta}(\cdot, t), \mathbb{R}) \leq C_1 T.V(u_0, \mathbb{R})$

(iii) $\int_{-\infty}^{\infty} |u_{\Delta x,\theta}(x, t) - u_{\Delta x,\theta}(x, \tau)| dx \leq C_5((t - \tau) + 6\Delta x) T.V(u_0, \mathbb{R})$

where $0 \leq \tau < t < \infty$ and C_1,C_2 and C_5 are constants that only depend on F.

Proof. The first part of the proof is based on [Gli65]. Choose a neighborhood $U_5 \subseteq U_4$ of 0_N, such that if $u_r, u_l \in U_5$ the range of the solution of the (u_l, u_r) Riemann problem is in U_4. Now we can choose C_4 and C_3 small enough, such that $u_{\Delta x,\theta}$ can be defined up to the line $t = \Delta t$ and the range of $u_{\Delta x,\theta}$ is in U_5. So we can define $u_{\Delta x,\theta}$ up to the line $t = 2\Delta t$ and the range of the approximative solution is in U_4. Also let C_4 be small enough, such that we can use the corollary from before. Thus again if C_3 and C_4 are small enough, the range of $u_{\Delta x,\theta}$ is in U_5. So we can define $u_{\Delta x,\theta}$ for all $t \leq 3\Delta t$. But, because of the corollary, the range of $u_{\Delta x,\theta}$ is again in U_5. So in that way we can define $u_{\Delta x,\theta}$ for all $t \geq 0$ and the first estimate holds for the approximative solution. Note that C_3 and C_4 do not depend on Δx or θ. So if C_3 and C_4 are small enough, approximative solutions can be defined for all Δx and θ.
The second estimate follows directly from the last theorem, if we choose $\tau_1 = 0$ and let $a \to -\infty$ and $b \to \infty$. To obtain the third estimate, according to [Daf16], we integrate the second estimate from last theorem over $(-\infty, \infty)$ and obtain

$$\int_{-\infty}^{\infty} |u_{\Delta x,\theta}(x, \tau_2) - u_{\Delta x,\theta}(x, \tau_1)| dx \leq \\ C_2 \int_{-\infty}^{\infty} T.V(u_{\Delta x,\theta}(\cdot, \tau_1), [x - c(\tau_2 - \tau_1) - 6\Delta x, x + c(\tau_2 - \tau_1) + 6\Delta x]) dx.$$

With Fubini's theorem we get

$$\begin{aligned} \int_{-\infty}^{\infty} |u_{\Delta x,\theta}(x, \tau_2) - u_{\Delta x,\theta}(x, \tau_1)| dx &\leq 2C_2(c(\tau_2 - \tau_1) + 6\Delta x) T.V(u_{\Delta x,\theta}(\cdot, \tau_1), \mathbb{R}) \\ &\leq 2C_1C_2(c(\tau_2 - \tau_1) + 6\Delta x) T.V(u_0, \mathbb{R}). \end{aligned}$$

That proves the third estimate with $C_5 = 2C_1C_2$. □

What we have presented is in fact a simplified version of the theorem originally proven in [Gli65]. In that theorem

$$||u_{\Delta x,\theta}(\cdot, t)||_\infty \leq K||u_0||_\infty$$

for some constant $K > 0$. This estimate can be obtained by defining another functional on mesh curves, like $\mathcal{F}$, which dominates the norm $||\cdot||_\infty$. This estimate is of interest since the total variation does not need to be small for the approximative solutions to be defined. But the estimate we obtained is enough to prove the existence of solutions.

3.4 Convergence of the Approximate Solutions

In this section the existence of weak solutions under some conditions will be shown, similarly to [Smo94]. We know from the last theorem, that there are constants C_3 and C_4, such that if the initial data of the initial value problem (1.1), (1.2) satisfies

(i) $||u_0||_\infty \leq C_3$

(ii) $T.V(u_0, \mathbb{R}) \leq C_4$

then the approximative solutions $u_{\Delta x,\theta}$ can be defined for all $\Delta x > 0$ and $\theta \in \mathcal{A}$. Lets assume u_0 satisfies the conditions above and let $\{u_{\Delta x,\theta}\}$ be the set of all approximative solutions.
Recall a function $u : \mathbb{R} \times \mathbb{R}_+ \to U$ is a weak solution of the initial value problem (1.1), (1.2) if

$$\int_0^\infty \int_{-\infty}^\infty u(x,t) \cdot \frac{\partial}{\partial t}\phi(x,t) + F(u(x,t)) \cdot \frac{\partial}{\partial x}\phi(x,t)dxdt + \int_{-\infty}^\infty u_0(x) \cdot \phi(x,0)dx = 0.$$

From the definition of the scheme we know an approximate solution $u_{\Delta x,\theta} \in \{u_{\Delta x,\theta}\}$ is a weak solution in every time strip $\{(x,t) \in \mathbb{R} \times \mathbb{R}_+ : n\Delta t \leq t < (n+1)\Delta t\}$, where $n \in \mathbb{N}$. So for each $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}_+)^N$ we have

$$\begin{aligned} 0 = &\int_{n\Delta t}^{(n+1)\Delta t} \int_{-\infty}^\infty u_{\Delta x,\theta}(x,t) \cdot \frac{\partial}{\partial t}\phi(x,t) + F(u_{\Delta x,\theta}(x,t)) \cdot \frac{\partial}{\partial x}\phi(x,t)dxdt \\ &+ \int_{-\infty}^\infty \phi(x, n\Delta t) \cdot u_{\Delta x,\theta}(x, n\Delta t)dx \\ &- \int_{-\infty}^\infty \phi(x,(n+1)\Delta t) \cdot u_{\Delta x,\theta}(x,(n+1)\Delta t-)dx, \end{aligned}$$

where

$$u_{\Delta x,\theta}(x,(n+1)\Delta t-) = \lim_{t \to (n+1)\Delta t} u_{\Delta x,\theta}(x,t).$$

It is important to note, that in general

$$u_{\Delta x,\theta}(x,(n+1)\Delta t-) \neq u_{\Delta x,\theta}(x,(n+1)\Delta t).$$

This is a result of how we defined the scheme. Now we can sum up the integrals from before to obtain

$$\begin{aligned} 0 = &\int_{-\infty}^\infty \int_{-\infty}^\infty u_{\Delta x,\theta}(x,t) \cdot \frac{\partial}{\partial t}\phi(x,t) + F(u_{\Delta x,\theta}(x,t)) \cdot \frac{\partial}{\partial x}\phi(x,t)dxdt \\ &+ \int_{-\infty}^\infty \phi(x,0) \cdot u_{\Delta x,\theta}(x,0)dx + \sum_{n=1}^\infty \int_{-\infty}^\infty \phi(x,n\Delta t) \cdot (u_{\Delta x,\theta}(x,n\Delta t) - u_{\Delta x,\theta}(x,n\Delta t-))dx. \end{aligned}$$

Define

$$\mathcal{J}_n(\theta, \Delta x, \phi) = \int_{-\infty}^\infty \phi(x,n\Delta t) \cdot (u_{\Delta x,\theta}(x,n\Delta t) - u_{\Delta x,\theta}(x,n\Delta t-))dx$$

and

$$\mathcal{J}(\theta, \Delta x, \phi) = \sum_{n=1}^\infty \mathcal{J}_n(\theta, \Delta x, \phi).$$

We will now show that there is a sequence $\Delta x_i \to 0$ for $i \to \infty$, such that $\mathcal{J}(\theta, \Delta x_i, \phi) \to 0$ for $i \to \infty$. To do so, we are going to need some properties of $\mathcal{J}(\theta, \Delta x, \phi)$. The first lemma is based on [Smo94] and the proof of the second one is taken from [Gli65].

Lemma 3.15. *Let $\theta \in \mathcal{A}$, $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}_+)^N$ and let $u_{\Delta x,\theta}$ be an approximate solution. Then there exist constants M_1 and M_2, such that*

$$|\mathcal{J}_n(\theta, \Delta x, \phi)| \leq M_1(\Delta x)||\phi||_\infty$$

and

$$|\mathcal{J}(\theta, \Delta x, \phi)| \leq M_2(\text{diam. support } \phi)||\phi||_\infty,$$

where (diam. support ϕ) *is the greatest distance between two points in the support of ϕ. The constants are independent of ϕ, θ and Δx.*

Proof. We will start with the first estimate. Let $(m, n) \in \mathcal{Y}$ and $\theta_{m,n} = (\theta^1_{m,n}, \theta^2_{m,n})$ a mesh point, then from the definition of the scheme we get

$$u_{\Delta x,\theta}(x, n\Delta t) = u_{\Delta x,\theta}(\theta^1_{m,n}, n\Delta t-).$$

for $(m-1)\Delta x \leq x \leq (m+1)\Delta x$. We can use this to obtain

$$\begin{aligned}
&\int_{(m-1)\Delta x}^{(m+1)\Delta x} |\phi(x, n\Delta t) \cdot (u_{\Delta x,\theta}(x, n\Delta t) - u_{\Delta x,\theta}(x, n\Delta t-))|dx \\
&\leq \int_{(m-1)\Delta x}^{(m+1)\Delta x} |\phi(x, n\Delta t) \cdot (u_{\Delta x,\theta}(\theta^1_{m,n}, n\Delta t-) - u_{\Delta x,\theta}(x, n\Delta t-))|dx \\
&\leq \int_{(m-1)\Delta x}^{(m+1)\Delta x} ||\phi||_\infty T.V(u_{\Delta x,\theta}(\cdot, n\Delta t-), [(m-1)\Delta x, (m+1)\Delta x])dx \\
&\leq 2\Delta x||\phi||_\infty T.V(u_{\Delta x,\theta}(\cdot, n\Delta t-), [(m-1)\Delta x, (m+1)\Delta x]).
\end{aligned}$$

Hence we get

$$\begin{aligned}
|\mathcal{J}_n(\theta, \Delta x, \phi)| &\leq \sum_{m=-\infty}^{\infty} 2\Delta x||\phi||_\infty T.V(u_{\Delta x,\theta}(\cdot, n\Delta t-), [(m-1)\Delta x, (m+1)\Delta x]) \\
&\leq 4\Delta x||\phi||_\infty T.V(u_{\Delta x,\theta}(\cdot, n\Delta t-), \mathbb{R}) \\
&\leq 4\Delta x C_1||\phi||_\infty T.V(u_0, \mathbb{R}) \\
&\leq M_1(\Delta x)||\phi||_\infty
\end{aligned}$$

where $M_1 > 0$ is a constant independent of ϕ, θ and Δx.
The support of ϕ is compact. So we can assume there are $a, b, T_1, T_2 \in \mathbb{R}$, such that the support of ϕ is in $[a, b] \times [T_1, T_2]$. Then $\mathcal{J}_n(\theta, \Delta x, \phi) \neq 0$ only if $T_1 \leq n\Delta t \leq T_2$. So there are at most

$$\frac{1}{\Delta t}(\text{diam. support } \phi) = (\frac{c}{\Delta x})(\text{diam. support } \phi)$$

nonzero summands in $\mathcal{J}(\theta, \Delta x, \phi)$. So we get

$$|\mathcal{J}(\theta, \Delta x, \phi)| \leq M_2(\text{diam. support } \phi)||\phi||_\infty$$

for some constant M_2 independent of ϕ, θ and Δx. □

Lemma 3.16. *Let $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}_+)^N$ and assume ϕ is piecewise constant on segments*

$$[(m-1)\Delta x, (m+1)\Delta x] \times \{n\Delta x\},$$

where $(m, n) \in \mathcal{Y}$. If $n_1 \neq n_2$, $n_1, n_2 \in \mathbb{N}$, then $\langle \mathcal{J}_{n_1}, \mathcal{J}_{n_2} \rangle_{L_2(\mathcal{A})} = 0$.

Proof. Let $(m_1,n_1),(m_2,n_2)\in\mathcal{Y}$ with $n_1<n_2$ and let

$$\mathcal{A}'=\mathcal{A}\backslash\{((m_2-1)\Delta x,(m_2+1)\Delta x)\times n_2\Delta t\}$$

and let $d\theta'$ be the product measure of $\mathcal{A}'$. The inner product $\langle\mathcal{J}_{n_1},\mathcal{J}_{n_2}\rangle_{L_2(\mathcal{A})}$ is a sum of terms

$$\int_{\mathcal{A}'}\int_{\mathcal{A}\backslash\mathcal{A}'}|\int_{(m_2-1)\Delta x}^{(m_2+1)\Delta x}\phi(x,n_2\Delta t)\cdot(u_{\Delta x,\theta}(x,n_2\Delta t)-u_{\Delta x,\theta}(x,n_2\Delta t-))dx|$$
$$|\int_{-\infty}^{\infty}\phi(x,n_1\Delta t)\cdot(u_{\Delta x,\theta}(x,n_1\Delta t)-u_{\Delta x,\theta}(x,n_1\Delta t-))dx|d\theta_{m_2,n_2}d\theta'$$

Note ϕ is constant on $[(m_2-1)\Delta x,(m_2+1)\Delta x]\times\{n_2\Delta x\}$ and

$$\int_{-\infty}^{\infty}\phi(x,n_1\Delta t)\cdot(u_{\Delta x,\Theta}(x,n_1\Delta t)-u_{\Delta x,\Theta}(x,n_1\Delta t-))dx$$

is independent of $d\theta_{m_2,n_2}$. We also have

$$\int_{\mathcal{A}\backslash\mathcal{A}'}\int_{(m_2-1)\Delta x}^{(m_2+1)\Delta x}u_{\Delta x,\theta}(x,n_2\Delta t)-u_{\Delta x,\theta}(x,n_2\Delta t-)dxd\theta_{m_2,n_2}$$
$$=\int_{\mathcal{A}\backslash\mathcal{A}'}\int_{(m_2-1)\Delta x}^{(m_2+1)\Delta x}u_{\Delta x,\theta}(\theta^1_{m_2,n_2},n_2\Delta t-)-u_{\Delta x,\theta}(x,n_2\Delta t-)dxd\theta_{m_2,n_2}=0$$

We can combine this to get

$$\int_{\mathcal{A}'}\int_{\mathcal{A}\backslash\mathcal{A}'}|\int_{(m_2-1)\Delta x}^{(m_2+1)\Delta x}\phi(x,n_2\Delta t)\cdot(u_{\Delta x,\theta}(x,n_2\Delta t)-u_{\Delta x,\theta}(x,n_2\Delta t-))dx|$$
$$|\int_{-\infty}^{\infty}\phi(x,n_2\Delta t)\cdot(u_{\Delta x,\theta}(x,n_1\Delta t)-u_{\Delta x,\theta}(x,n_1\Delta t-))dx|d\theta_{m_2,n_2}d\theta'=0$$

So we get $\langle\mathcal{J}_{n_1},\mathcal{J}_{n_2}\rangle_{L_2(\mathcal{A})}=0$. □

If $\Delta x=2^{-k}$ for $k\in\mathbb{N}$ and ϕ is piecewise constant on segments $[(m-1)\Delta x,(m+1)\Delta x]\times\{n\Delta x\}$, where $(m,n)\in\mathcal{Y}$. Then ϕ is piecewise constant on these segments for all $\Delta x=2^{-q}$, where $q\in\mathbb{N}$ with $q\geq k$.
According to [Gli65], there is an isomorphism of $\mathcal{A}$ with

$$\prod[0,1],$$

an infinite product of copies of $[0,1]$. The isomorphism is given by

$$\theta_{m,n}\mapsto\frac{1}{2}(\theta^1_{m,n}\frac{1}{\Delta x}-m+1)$$

for all $(m,n)\in\mathcal{Y}$. Therefore we can consider $\mathcal{A}$ to be independent of Δx. This is important for the next theorem, which is based on [Smo94].

Theorem 3.17. *Let $\{u_{\Delta x,\theta}\}$ be the set of all approximative solutions like before. Then there exists a null set $\mathcal{N}\subset\mathcal{A}$ and a sequence $\Delta x_i\to 0$ as $i\to\infty$, such that $\mathcal{J}(\theta,\Delta x_i,\phi)\to 0$ as $i\to\infty$ for any $\theta\in\mathcal{A}\backslash\mathcal{N}$ and any function $\phi\in C^1_0(\mathbb{R}\times\mathbb{R}_+)^N$.*

Proof. Let $\Delta x_i=2^{-i}$, $i\in\mathbb{N}$, and let $\phi\in C^1_0(\mathbb{R}\times\mathbb{R}_+)^N$ be piecewise constant on segments $[(m-1)\Delta x_k,(m+1)\Delta x_k]\times\{n\Delta t\}$, where $(m,n)\in\mathcal{Y}$ and $k\in\mathbb{N}$. Then for all $i\geq k$ we get

$$||\mathcal{J}(\cdot,\Delta x_i,\phi)||_2^2=\sum_{n=-\infty}^{\infty}||\mathcal{J}_n(\cdot,\Delta x_i,\phi)||_2^2+\sum_{q,p=-\infty,q\neq p}^{\infty}\langle\mathcal{J}_q,\mathcal{J}_p\rangle_{L_2(\mathcal{A})}$$
$$=\sum_{n=-\infty}^{\infty}||\mathcal{J}_n(\cdot,\Delta x_i,\phi)||_2^2.$$

Also we have

$$\sum_{n=-\infty}^{\infty} ||\mathcal{J}_n(\cdot, \Delta x_i, \phi)||_2^2 = \sum_{n\in\Lambda} \int_{\mathcal{A}} |\mathcal{J}_n(\cdot, \Delta x_i, \phi)|^2 d\theta$$
$$\leq \sum_{n\in\Lambda} \int_{\mathcal{A}} |M_1(\Delta x_i)||\phi||_\infty|^2 d\theta$$
$$\leq \sum_{n\in\Lambda} |M_1(\Delta x_i)||\phi||_\infty|^2,$$

where

$$\Lambda = \{n \in \mathbb{N} : (m, n) \in \mathcal{Y} \text{ and } \phi(x, n\Delta t) \neq 0 \text{ for some } x \in \mathbb{R}\}.$$

So is follows

$$||\mathcal{J}(\cdot, \Delta x_i, \phi)||_2^2 \leq M_1^2 \Delta x_i (\text{diam .support } \phi) ||\phi||_\infty^2.$$

So for each $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}_+)^N$, that is piecewise constant on segments

$$[(m-1)\Delta x_p, (m+1)\Delta x_p] \times \{n\Delta t\}$$

where $(m, n) \in \mathcal{Y}$ and $\Delta x_p = 2^{-p}$ for some $p \in \mathbb{N}$, there exists a sequence (Δx_i) with $\lim_{i\to\infty} \Delta x_i = 0$, such that $\lim_{i\to\infty} ||\mathcal{J}(\cdot, \Delta x_i, \phi)||_2 = 0$.
Now let $(\phi_k) \subset C_0^1(\mathbb{R} \times \mathbb{R}_+)^N$ be a sequence of piecewise constant functions like above, which are dense in $C_0^1(\mathbb{R} \times \mathbb{R}_+)^N$. For every $k \in \mathbb{N}$ there exists a null set $\mathcal{N}_k \subset \mathcal{A}$ and a sequence $(\Delta x_i^k) \subset \mathbb{R}$, such that

$$\mathcal{J}(\theta, \Delta x_i^k, \phi_k) \to 0$$

as $i \to \infty$ if $\theta \in \mathcal{A} \backslash \mathcal{N}_k$. By a standard diagonal process we can find a subsequence, for simplicity let us call the subsequence (Δx_i) again, such that

$$\mathcal{J}(\theta, \Delta x_i, \phi_k) \to 0$$

as $i \to \infty$ for all $\theta \notin \bigcup_{j\in\mathbb{N}} \mathcal{N}_j$ and $k \in \mathbb{N}$. Let $\mathcal{N} = \bigcup_{j\in\mathbb{N}} \mathcal{N}_j$.
Now let $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}_+)^N$ not necessarily piecewise constant, then for any $\theta \in \mathcal{A} \backslash \mathcal{N}$ and $k \in \mathbb{N}$ we get

$$|\mathcal{J}(\theta, \Delta x_i, \phi)| \leq |\mathcal{J}(\theta, \Delta x_i, \phi - \phi_k)| + |\mathcal{J}(\theta, \Delta x_i, \phi_k)|.$$

Let $\varepsilon > 0$, then we can choose $k \in \mathbb{N}$ in such way, that

$$|\mathcal{J}(\theta, \Delta x_i, \phi - \phi_k)| < \frac{1}{2}\varepsilon.$$

This follows from the first lemma. Then we can choose i so large, that

$$|\mathcal{J}(\theta, \Delta x_i, \phi_k)| < \frac{1}{2}\varepsilon.$$

This can be done by our previous findings. So we get

$$|\mathcal{J}(\theta, \Delta x_i, \phi)| < \varepsilon.$$

So for any $\theta \in \mathcal{A} \backslash \mathcal{N}$ and any function $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}_+)^N$ we have $\mathcal{J}(\theta, \Delta x_i, \phi) \to 0$ as $i \to \infty$. □

Let (Δx_i) be the sequence obtained in the last theorem, then set $u_i = u_{\Delta x, \theta}$ for any $\theta \in \mathcal{A} \backslash \mathcal{N}$. Then the following theorem from [Smo94] states

Theorem 3.18. *Let (u_i) be a sequence of approximative solutions, like above, then there exists a subsequence (u_i'), that converges in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_+)^N$ to a function u.*

We will present the idea of the proof. Since the sequence of approximative solution is bounded and has bounded total variation we can use Helly's theorem [PM91], to find a subsequence, that converges at each point on any bounded interval of any given horizontal line in the x-t-plane. By a standard diagonal process we can construct another subsequence, that converges on every point $x \in \mathbb{R}$ for a $t > 0$. By choosing a dense, countable subset of an interval $(0, T)$ with $T > 0$ one can show with the third estimate for the approximative solution, that the subsequence is Cauchy in $L_1(\{x \in \mathbb{R} : |x| \leq X\} \times (0, T))^N$ for any $X > 0$. This can be used to show, that the subsequence, call it (u'_n), converges to a function u in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_+)^N$. This completes the idea of the proof.

Now let (u'_n) be the sequence obtained in the last theorem. According to [Smo94] there exists a subsequence (u''_n), which converges to the u almost everywhere. Since F is smooth $F(u''_n)$ converges to $F(u)$ almost everywhere. We also know, that the sequence $F(u''_n)$ is uniformly bounded, so with the dominated convergence theorem [EG15] we get $F(u''_n) \to F(u)$ for $n \to \infty$ in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_+)^N$. With that we get

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x,t) \cdot \frac{\partial}{\partial t}\phi(x,t) + F(u(x,t)) \cdot \frac{\partial}{\partial x}\phi(x,t) dx dt + \int_{-\infty}^{\infty} \phi(x,0) \cdot u(x,0) dx = 0.$$

But from the definition of our scheme follows, according to [Gli65], that $u'_n(x,0) \to u_0(x)$ for $n \to \infty$ in $L^1_{loc}(\mathbb{R})$. So we obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x,t) \cdot \frac{\partial}{\partial t}\phi(x,t) + F(u(x,t)) \cdot \frac{\partial}{\partial x}\phi(x,t) dx dt + \int_{-\infty}^{\infty} \phi(x,0) \cdot u_0(x) dx = 0.$$

So the limit function u is indeed a weak solution of the initial value problem (1.1), (1.2). So we have shown, that under certain conditions weak solutions of the initial value problem (1.1), (1.2) exists. We will end this section by stating our findings in the following theorem.

Theorem 3.19. *Assume we are given an initial value problem 1.1, 1.2. There exists positive constants C_3 and C_4, such that if*

(i) $||u_0||_\infty \leq C_3$

(ii) $T.V(u_0, \mathbb{R}) \leq C_4$

a weak solution of the initial value problem (1.1), (1.2) *exists.*

4 Conclusion and Outlook

In this thesis, a way to construct weak solutions of the initial value problem for general systems of conservation laws in one space dimension using Glimm's scheme is presented. The particular aim is to focus on the most essential parts published in different references (e.g. [Smo94], [Daf16], [GR21] or [Eva98]). In the first chapter the system of equation and initial value problem is defined and weak solutions are introduced. Subsequently, it is shown, that the Riemann problem for one-dimensional systems of conservation laws has a weak solution if the initial data satisfies certain conditions. Finally Glimm's scheme is introduced. Based on that scheme a way to show the existence of weak solutions of the initial value problem is presented.

As a conclusion, based on the considered references, this thesis presents the essential basics and results, which are required to show the existence of solutions of the initial value problem for one-dimensional systems of hyperbolic conservation laws, in a more focused and holistic way.

James Glimm developed this scheme and thereby the existence proof in 1965. As we stated in chapter 3, the estimates we obtained are a simplified version of the estimates in [Gli65]. This estimates were sufficient to show the existence, but are more restrictive, than the original estimates. So it would be of interest to present the proof of the more general estimates.
Also numerous new results were published over the last decades. For example the convergence of the scheme we presented depends on choosing a random sequence $\theta \in \mathcal{A}$. Tai-Ping Liu showed in [Liu77], that this is not necessary, since the scheme converges for any equidistributed sequence. The ideas of the scheme can be used to develop a numerical method as Alexandre Chorin did in [Cho76]. Also an alternative way to prove the existence of solutions to systems of hyperbolic conservation laws was shown by Nils Hendrik Risebro in [Ris93].

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